LOGICA 2025

ABSTRACTS

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Computational learning and dynamic logic: New perspectives

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The ability to learn is an essential component of autonomous agency and, as such, is a ubiquitous topic of study in formal epistemology and Artificial Intelligence. The design of computational learners has led to a range of practical solutions that are now applied across various real-life domains. Whether a computational learning process has been successful is often treated as an empirical question—the performance of different learning algorithms is compared using specific benchmark problems or evaluated against human performance.

Nevertheless, a more abstract line of research persists, addressing the question of more robust, theoretical guarantees. In my talk, I will present such perspectives, focusing particularly on those based on dynamic modal logic: learning as transitioning between nodes in a graph of hypotheses (Gierasimczuk et al., 2009), learning by leveraging observations in epistemic and topological spaces (Baltag et al., 2019, 2015; Gierasimczuk, 2023), and learning as propagation in a (neural) network (Kisby et al., 2024). Finally, I will discuss the potential impact of these logic-based accounts on Knowledge Representation and explainability in Artificial Intelligence.

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What are support conditionals?

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In natural language, conditionals are frequently used for giving explanations. The antecedent of a conditional is typically understood as providing evidential support for (being connected to, being relevant for, or making a difference to) the conditional's consequent. This aspect has not been reflected in the logics that are usually offered for the reasoning with conditionals: neither in the logic of the material conditional or the strict conditional, nor in the plethora of logics for suppositional conditionals that have been produced over the past 50 years. In this talk I survey some recent attempts to come to terms with the problem of encoding evidential support in the logic of conditionals. I have a look at resulting logical properties and related inferentialist, connexivist and quantitative-probabilistic ideas.

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Knowledge and belief: Between epistemic logic and theory of mind

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Epistemic logic, the logic of knowledge, appears to be crucial in describing negotiations in economics, parallel processors in computer science, and multi-agent systems in artificial intelligence. One of the main questions of the lecture will be: In which ways is epistemic logic an idealization of human social reasoning? How do people actually reason about their own and other people's knowledge and beliefs, both in story situations where different participants have different perspectives as well as in competitive games and negotiations? We will report on several experiments about this so-called 'theory of mind', with both children and adults: While the usefulness of higher orders of theory of mind is apparent in many social interactions, empirical evidence so far suggests that people usually do not use this recursive ability spontaneously, even when doing so would be highly beneficial. Moreover, we will discuss how a variation of dynamic epistemic logic can be useful as a guide for experiments and computational cognitive modeling. Finally, we will discuss the theory-of-mind abilities of ChatGPT.

Sequentiality via Markov coding

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Gödel's original proof of the sequentiality of arithmetic is still one of the best ways to build sequences over arithmetic with addition and multiplication. Emil Jeřábek shows that we already have the beta-function with its good properties in a weak arithmetic that is even weaker than the usual base theory PA^-. There are however other ways of making sequences. Two strategies are based on translating (binary) strings into arithmetic. The first one is due to Raymond Smullyan and the second one is due to Andrej Markov jr. Both coding strategies are strikingly different.

These alternative strategies are interesting both for didactical reasons and for philosophical ones. We will briefly comment on this.

In this talk we discuss the Markov strategy. The basic idea is that the special linear monoid of $\mathbb{N}$ is isomorphic to the binary strings. We show that the Markov strategy can be made to work in PA^- plus the Euclidean Division Axiom.

We discuss two recursive models of PA^- and reflect on what the Markov strings in these models look like. The results suggest that there is a mystery extension of PA^- that is better for our purpose than our extension with Euclidean Division, but currently I have no good proposal on what it might be.

Contributed talks

Contradictory logics are queer feminist logics

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Feminist logic is an area of study that seems underrepresented both in logic and in feminist philosophy.[1] The reason for the former is that many (most?) logicians simply do not see any connection or applicability of logic to feminist issues. The reason for the latter is that there is some feminist literature, e.g., (Nye, 1990), arguing that feminism and logic are in principle incompatible. However, proving both sides wrong in practice, feminist logic has developed as a small but upcoming discipline.[2] Feminist logic is often used as a short form including both what can be understood as 'feminist logic' and 'feminist philosophy of logic'. These two areas are very much intertwined, usually informing, affecting or guiding each other, which is why a strict, clear-cut distinction is not needed but rather we can give some examples of what these can/do include: Feminist logic may be conducted by formalizing notions that are especially important for feminist reasoning (Saint-Croix & Cook, 2024) or by applying logic(s) to feminist ends (Russell, 2024), or by devising, revising and/or arguing for logical systems from a distinctly feminist perspective.
Within the latter approach one can concentrate on analyzing logical systems with respect to structural features that may perpetuate sexism and oppression or, on the other hand, features that may be helpful for resisting and opposing these social phenomena. Arguably the most central work here is Val Plumwood's (1993) feminist critique of classical logic, which will be the starting point for my own considerations. Plumwood sees classical logic as a "Logic of Domination" by implementing and perpetuating what she calls "dualisms", a special kind of dichotomies resulting from and simultaneously yielding the domination of one concept over the other. This is said to be established especially by the conception of classical negation when $\sim p$ is interpreted as 'the other of p' in a hierarchical, dualistic way. To clarify the points of criticism that are most significant for the present purpose, I will only mention three of the five features characteristic of dualisms that she claims to be inherent in classical logic and thus, to be responsible for a 'naturalization of domination', resulting from the (often supposed) universality and ubiquity of classical logic.

Relational Definition (*Incorporation*): The other (e.g., 'women') is not defined in its own terms or positively but completely in dependence on the dominant side of the dualism as a lack or negativity (e.g., as 'not men').

Radical Exclusion (*Hyperseparation*): In a dualistic relationship the other is not only treated as different but as inferior, and to that end number and importance of differences between the sides is overemphasized by the dominant group and a possible overlap is denied. The dualistic pairs are constructed complementary, having "characteristics which exclude but logically require a corresponding and complementary set in the other" (Plumwood, 1993, p.

[1] Feminism can be broadly understood as the socio-political movement that aims to establish social, political, economic and personal gender equality. I use the term 'queer feminist' here also in its very broad sense according to which the perspective is taken that, firstly, gender and sexuality are central to any understanding of wider social and political processes, and secondly, these categories are to be studied as intersecting with other social inequalities like racialization, economic status, disabilities, etc. Since this is a rather recent development in the feminist debates, when referring to older literature I will only use 'feminist'.

[2] See, e.g., Eckert and Donahue (2020); Ferguson (2023); Mangraviti (2023); Russell (2024); Saint-Croix and Cook (2024).

449). The logical principle reflecting this is in Plumwood's opinion the principle of explosion: p and its other ($\sim p$) are to be kept at maximum distance; bringing them together yields the worst-case scenario of system collapse.

Homogenisation (*Stereotyping*): To confirm the 'nature' of the dualistic pairs both the dominant group as well as the dominated must appear maximally homogeneous. Therefore, stereotyping is used as an instrument of domination, whereby similarities are overemphasized, while differences within these groups are disregarded. There is no room for differentiation, everything other-than-p must fall under $\sim p$.

What I want to argue for is that from this point of view so-called *contradictory logics*, i.e., non-trivial systems containing contradictions in their set of theorems, are worthwhile to consider both because of their structural features and because of the important role that contradictions play in queer feminist theories. I want to show that, on the one hand, the formal set-up of contradictory logics – especially when a bilateralist representation is used – makes them well-suited from the perspective of feminist views on logic and, on the other hand, that queer feminist theories provide a relevant, and so far undeveloped, conceptual motivation for contradictory logics. Therefore, while using the connexive, contradictory logic $\mathtt{C}$ (Wansing, 2005) here as an example to explain these points further and to motivate my account, I will deviate from its (and other contradictory logics') usual representation by considering a notion of contradiction that does *not* need to rely on negation as an underlying concept.

$\mathtt{C}$ is a contra-classical logic in that it validates theorems classical logic does not have. Thus, unlike most alternative (paraconsistent) logics considered by feminist logicians, in this case we have a logic which is not even a subsystem of classical logic. If we do consider Plumwood's criticism of classical logic valid, it seems desirable to free ourselves as rigidly as possible from it. Contradictory logics do that by going beyond paraconsistency in that they are not only not explosive but actually have *contradictory theorems*, in the sense that there are formulas A for which there is both a proof of A and a proof of $\sim A$. On a bilateralist account of a contradictory logic like $\mathtt{C}$, though, we can get rid of (at least a primitive account of) negation completely. Specifically, this can be done by considering two derivability relations instead. Proof-theoretic bilateralism takes two dichotomic concepts, traditionally the speech acts of assertion and denial, strictly on a par and not one as reducible to the other. Here, instead of speech acts, I will rather consider the concepts of proof and refutation and show how these can be implemented proof-theoretically. Instead of conceiving contradictions in terms of negation, as it is most usually done,[3] we can then have contradictions by having A both provable and refutable in our system. I will show how such a bilateralist contradictory system provides a dichotomic, yet not dualistic (in the Plumwoodian negative sense), representation of a logic. As a suitable system for this I will consider the negation-free fragment of the bi-connexive logic $\mathtt{2C}$ as developed in (Wansing, 2016) and show some features that are the outcome of dismissing strong negation. I will show how this logic 1) can avoid the dualistic features of Relational Definition, Radical Exclusion and Homogenisation and 2) constitutes a prime example of a desideratum implicit (and often also explicit) in queer feminist theories: to *accommodate* contradictions instead of trying to avoid or overcome them.

It is the second point that not only makes contradictory logics interesting for queer feminist reasoning but also the other way around: it makes queer feminist theories interesting for contradictory logics because they constitute actual examples from philosophy of science, epistemology, etc. which explicitly endorse the existence of contradictions. Importantly, this

[3] Also by Plumwood, see, e.g., comments like "contradiction being parasitic on negation" (p. 201) or "Contradiction is always characterized in terms of negation and the logical behaviour of contradictions is dependent on that of negation" (p. 204) in (Routley & Routley, 1985).

happens on two levels: the theories are contradictory *and* the world itself is contradictory; the latter situation essentially being the cause for the former. I will show examples of such contradictions, explain why they play such an integral role for queer feminist theories and why they are *not* considered problematic in these contexts but rather welcomed as a tool for navigating in a patriarchal society. Such a reasoning about social dimensions being contradictory has not received much attention in the area of formal logic despite there being a great interest in contradictions, paradoxes, etc. This lack of attention is in my opinion mainly due to the contingent fact that feminist philosophy (and social philosophy in general) has had a historically close connection to continental philosophy, i.e., to an area famously disregarding formal methods. I do not see any essential reason, though, why these two areas should exclude each other.

Investigating whether and to what extent Plumwood's desiderata are met by this account, though, will show that we have good reasons to want to go further than that. As the "bi" in "bilateralism" clearly tells us, there is a *binarism* inherent in that picture. While Plumwood's remarks seem ambiguous on whether or not this aspect is to be retained or should be overcome, nowadays a strictly binary view with respect to concepts of gender but also others like race, sexuality, disability, etc. seems unsuitable. What seems rather appropriate here is to consider a wide spectrum accommodating fluidity for these concepts. Thus, as a tentative outlook I would like to consider whether a conception of *multilateralism*, as e.g. developed in (Wansing & Ayhan, 2023), might provide a useful account for tackling this problem.

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Towards a proof-theoretic analysis of incorrect proofs

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Many mathematical discoveries were stated together with incorrect proofs (maybe the most famous example being the first attempt to solve the Basel problem by Leonard Euler (Sandifer, 2003), (Lecat, 1935)), therefore, two questions arise:

1. If the proof is reasonable, the statement is true, but the proof is incomplete. What information is needed to complete the proof?

2. If the proof is reasonable, the statement is not true, and the proof is (necessarily) incorrect/incomplete. What is a reasonable weakened statement that can be proved by (a close variant) of the given proof?

The two questions are dual as an incomplete proof of a somewhat weakened statement A' completed by B leads to a correct proof of $B \rightarrow A'$.

In this lecture, it is intended to investigate formal proofs within a logical framework and how to determine the minimal (weakest) preconditions to make a proof correct (weakest in the sense that all possible preconditions imply the minimal condition).

Gödel's incompleteness theorems allegedly show the impossibility of Hilbert's programme as such but thus have stood and mark the beginning of a seminal paradigm shift in proof theory: to use mathematical proofs as a rich source of computational information such as certified algorithms and effective bounds. This lecture belongs to this specific area of proof theory.

In classical propositional logic, the non-validity of a formula can be identified with its incorrect proof: the minimal information to make, e.g., $A \wedge B \rightarrow A \wedge C$ valid is $\neg(A \wedge B \wedge \neg C)$ excluding the only line of the truth table falsifying the formula. This means

$$\neg(A \wedge B \wedge \neg C) \rightarrow (A \wedge B \rightarrow A \wedge C)$$

is a tautology ($\perp$ is the weakest precondition if the formula is already valid).

However, what to do with first-order expressions? To determine the minimal lacking information to make the expression valid is undecidable as "$\perp$ is the minimal information to make A valid" is equivalent to "A is valid". Furthermore, such a weakest information might not exist.

$$\forall x(A(p(x)) \rightarrow A(x)) \rightarrow A(0)$$

has using Herbrand's theorem the valididating premises $A(0), A(p(0)), A(p(p(0))), \ldots$ and $A(p^{n+1}(0))$ is weaker then $A(p^n(0))$.

In first-order logic, we have to consider specific incorrect proofs, maybe in sequent calculus instead of incorrect formulas

Example 1. $\forall x(A(x) \wedge \exists y B(y)) \rightarrow \forall x A(x) \wedge \exists x C(x)$.

$$
\begin{array}{cc}
A(a) \Rightarrow A(a) & \exists y B(y) \Rightarrow \exists y B(y) \\ \hline
A(a) \land \exists y B(y) \Rightarrow A(a) & \exists y B(y) \Rightarrow \exists y C(y) \quad * \\ \hline
\forall x(A(x) \land \exists y B(y)) \Rightarrow A(a) & A(a) \land \exists y B(y) \Rightarrow \exists y C(y) \\ \hline
\forall x(A(x) \land \exists y B(y)) \Rightarrow \forall x A(x) & \forall x(A(x) \land \exists y B(y)) \Rightarrow \exists y C(y) \\ \hline
\multicolumn{2}{c}{\forall x(A(x) \land \exists y B(y)) \Rightarrow \forall x A(x) \land \exists y C(y)} \\ \hline
\multicolumn{2}{c}{\Rightarrow \forall x(A(x) \land \exists y B(y)) \to \forall x A(x) \land \exists y C(y)}
\end{array}
$$

The obvious correction of the error is obtained from $\exists x B(x) \to \exists y C(y)$, but the argument is also valid if $\neg\forall x A(x)$ holds. $\neg\forall x A(x) \lor (\exists y B(y) \to \exists y C(y))$ is the guessed approach to a weakest premis. But how to calculate this in general? This lecture intends to provide a solution to this problem based on ε-calculus. We focus on incorrect mathematical proofs where each single quantifier inference is correct, and therefore the proofs can be written in regular tree form.

ε-calculus (Hilbert, 1939),(Zach, 2006),(Baaz, 2022) is based on the replacement of $\exists x A(x)$ by $A(\varepsilon_x A(x))$ and $\forall x A(x)$ by $A(\varepsilon_x \neg A(x))$.

Example 2. *The standard translation of* $\exists x A(x) \land \forall x B(x)$ *is* $A(\varepsilon_x A(x)) \land B(\varepsilon_x \neg B(x))$.

LK-proofs are translated to ε-calculus by replacing $\exists$-right and $\forall$-left by $A(t) \to A(\varepsilon_x A(x))$ and $\neg A(t) \to \neg A(\varepsilon_x \neg A(x))$ on the left side of the sequent. $\exists$-left and $\forall$-right are replaced by substitution of the corresponding ε-terms and discharged. The resulting sequent is a tautology, which is an ε-proof in itself; therefore, it can be handled as in the propositional case.

Example 3. $e_1 \equiv \varepsilon_x \neg A(x)$
$e_2 \equiv \varepsilon_y B(y)$
$e_3 \equiv \varepsilon_y C(y)$
$e_4 \equiv \varepsilon_x \neg(A(x) \land B(e_2))$

$$
\begin{array}{cc}
 & B(e_2) \Rightarrow B(e_2) \\ \hline
A(e_1) \Rightarrow A(e_1) & B(e_2) \Rightarrow C(e_3) \quad * \\ \hline
A(e_1) \land B(e_2) \Rightarrow A(e_1) & A(e_1) \land B(e_2) \Rightarrow C(e_3) \\ \hline
A(e_4) \land B(e_2) \Rightarrow A(e_1) & A(e_4) \land B(e_2) \Rightarrow C(e_3) \\ \hline
\multicolumn{2}{c}{A(e_4) \land B(e_2) \Rightarrow A(e_1) \land C(e_3)} \\ \hline
\multicolumn{2}{c}{\Rightarrow A(e_4) \land B(e_2) \to A(e_1) \land C(e_3)}
\end{array}
$$

The ε*-proof is*

$$\neg(A(e_1) \land B(e_2)) \to \neg(A(e_4) \land B(e_2)) \to A(e_4) \land B(e_2) \to A(e_2) \land C(e_2)$$

The weakest precondition is calculated from the truth table

$$\neg(A(e_1) \land B(e_2)) \to C(e_3)$$

which retranslated to first-order language is

$$\neg\forall x(A(x) \land \exists y C(y)) \lor \exists y B(y)$$

which is equivalent but not identical to the guessed weakest premise

$$\neg\forall x A(x) \lor (\exists x B(x) \to \exists x C(x))$$

Note that this retranslation is not trivial and that no general reasonable algorithm is known. In this lecture, we will develop proof-theoretic tools based on ε-calculus to analyse incorrect proofs.

As an example, we provide the following theorem about ε-calculus.

Theorem 1. *The algorithm of the extended first ε-theorem is false-tolerant: if there is at most only one interpretation that falsifies the proof (i.e., one line of the minimal truth table) then the same holds for the Herbrand disjunction obtained after elimination of critical formulas.*

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A logical framework for data-driven inference

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This work provides a general logical framework for modeling patterns of reasoning with data, in particular when data may be uncertain or misleading. As a particularly problematic example of a pattern of reasoning with data in science, consider null hypothesis significance testing (NHST) in statistical inference. NHST and its (mis)-application has raised heated controversies in the last years (13).

Let H_0 be a sentence (in classical logic) standing for what is usually referred to as the *null hypothesis*. This is the hypothesis that no effect exists, concerning a particular phenomenon. The key idea in NHST is thus to set up an experiment, collecting data with the aim of *rejecting* H_0. In NHST, one associates to the observed data a quantity, usually referred to as the "p-value". This latter is the calculated conditional probability of seeing equally extreme or more extreme data, under the assumption that H_0 holds. In analogy with modus tollens, one says that H_0 should be rejected if the p-value is small enough.

In synthesis, the argument goes as follows: 1) Suppose H_0; 2) Calculate the p-value for some test statistics, i.e. a well-defined function of the data observed conditional on H_0; 3) The smaller the p-value, the stronger the reason to believe that H_0 is not true.

Many statisticians and methodologists explicitly draw the parallel between (versions of) the above and modus tollens (2; 7; 11) and, quite interestingly, this view is shared also by prominent critics of NHST, e.g. (12).

The analogy between NHST and modus tollens is appealing, but shallow, since no probabilistic test can ever deliver $\neg H_0$ in the sense of classical logic. We find this to be a source of confusion in the literature, and propose to frame this and other patterns formally, by use of suitable non-classical logics. Although non-classical logics are mostly unknown to the statisticians and science methodologists, we believe that they have much unexplored potential (few notable exceptions being e.g. (1; 5)), at least in clarifying the different positions and approaches.

In particular, our proposal is to frame patterns of data-driven inferences under the lens of non-monotonic logics (6), developed within the AI community. We provide a blue-print non-monotonic consequence relation to model inferences from data to hypotheses, that satisfy the rules of *rational* consequence relations, the so-called System R in the non-monotototonic literature. Our system departs, however, from system R, in several respects.

First, we distinguish between a (propositional) language for data, with set of formulas $\mathrm{Fm}_{\mathscr{D}}$, based on the propositional variables $\mathscr{D} = \{d_1, \ldots, d_l\}$, and a language for hypotheses,

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with formulas $\mathrm{Fm}_{\mathscr{H}}$ based on propositional variables $\mathscr{H} = \{H_1, \dots, H_n\}$. We assume that a theory $T \subseteq \mathrm{Fm}_{\mathscr{H}}$ expresses background non-revisable knowledge, e.g. stating that the hypotheses are mutually exclusive and exhaustive.

Second, since experiments (the data-generating processes) can be repeated, and hypotheses may be confronted multiple times by the same data, we opt for using as premises in our consequence relation *multisets* of formulas for data. We denote the multisets of formulas in $\mathrm{Fm}_{\mathscr{D}}$ by $\mathscr{M}_{\mathscr{D}}$.

Finally, we center the semantics on the notion of *the degree to which a piece of data $\delta \in \mathrm{Fm}_{\mathscr{D}}$ rejects a hypothesis $\varphi \in \mathrm{Fm}_{\mathscr{H}}$*. This is defined as follows (where $\models$ denotes the classical consequence relation).

Definition 1. We say that $r \colon \mathscr{M}_{\mathscr{D}} \times \mathscr{H} \to \mathbb{R}_+ \cup \{\infty\}$ is a *degree of rejection* if, for any $\gamma, \delta \in \mathrm{Fm}_{\mathscr{D}}$, $\Delta \in \mathscr{M}_{\mathscr{D}}$ and $H \in \mathscr{H}$, the following hold:

1. If $T, \gamma \models \delta$, then $r_\gamma(H) \geq r_\delta(H)$.
2. If $T, H \models \delta$ then $r_\delta(H) = 0$.
3. If $T, \delta \models \neg H$, then $r_\delta(H) > 0$.
4. If $r_\Delta(H) \neq \infty$, then $r_\Delta(H) = \sum_{\delta \in \Delta} r_\delta(H)$.

Given a probability function P over $\mathrm{Fm}_{\mathscr{D}} \cup \mathrm{Fm}_{\mathscr{H}}$. two prominent and motivating examples of rejection functions are obtained by setting, for instance

$$r_\delta(H) = P(\neg\delta | H) \qquad r_\delta(H) = -logP(\delta | H).$$

The notion of degree of rejection allows us to define a basic blueprint consequence relation as follows. Let $\check{\mathscr{H}}_\Delta$ be the set of hypotheses in $\mathscr{H}$ which are *least rejected* by the data in Δ, *i.e.*:

$$\check{\mathscr{H}}_\Delta = \operatorname*{arg\,min}_{H \in \mathscr{H}^f_\Delta} r_\Delta(H). \tag{1}$$

where $\mathscr{H}^f_\Delta$ is the set of hypotheses with a finite degree of rejection.

We may thus define a blueprint consequence $\mid\!\sim_{RJ}$ as classical model-preservation under least-rejection.

Definition 2 (RJ-consequence)**.** Let $\Delta \in \mathscr{M}_{\mathscr{D}}$ and $\varphi \in \mathrm{Fm}_{\mathscr{H}}$. We say that φ is an *RJ-consequence of* Δ, written $\Delta \mid\!\sim_{RJ} \varphi$, if $T, H \models \varphi$, for each $H \in \check{\mathscr{H}}_\Delta$.

We investigate different instances of rejection degrees and the rules that the corresponding consequence relation satisfy. In particular, when the rejection degrees are instantiated probabilistically, we obtain consequence relations that are closely related to maximum likelihood and NHST. Such consequence relations satisfy, in addition to the rules of rational consequence relation, a peculiar form of conjunction rule on the left, which is not satisfied in general by the blueprint consequence $\mid\!\sim_{RJ}$.

A final interesting instance of the rejection function and of a corresponding consequence relation is obtained by considering the so-called Ulam games. We can think of such games as played by *Scientists* (**S**) against *Nature* (**N**). **N** "thinks" of a number, which is the index of the unique true hypothesis in $\mathscr{H} = \{H_1, \dots, H_n\}$. **S** must figure out which one is such hypothesis and aims to do so as quickly as possible. The only move available to **S** is to ask Nature binary questions, i.e. questions that can be answered with either "Yes" or "No". In this latter case we say that *the hypothesis has been rejected.* While standard Ulam games provide a sound a complete semantics for classical logic, a generalized version of the game allowing allowing N may *lie m* times ($m \in \mathbb{N}$) provides a sound and complete semantics for the $(m+2)$-valued Łukasiewicz logic (8).

We argue that these games have a direct link with our running theme of modeling patterns of reasoning with data, in particular what is known in the literature as *strong inference* (9), a generalization of NHST where emphasis is put on considering multiple hypotheses and pruning them by way of empirical rejection. We may thus interpret lies in Ulam games in terms of misleading data, due to mistakes made by scientists involved in the process of strong inference, either in the design of experiments or in the analysis of the data generated by them.

We then define a rejection degree based on rejection in Ulam games and provide a consequence relation $\mid\!\sim_{uRJ}$ on the model of our blueprint $\mid\!\sim_{RJ}$, where the premises express a given stage of the game and the conclusion the minimally rejected hypotheses at that stage. We show that the system satisfies, in addition to the usual properties of $\mid\!\sim_{RJ}$ and rational consequence relations, a form of constrained monotonicity which, to the best of our knowledge, has not been investigated before. The main technical result of our work is a completeness result for $\mid\!\sim_{uRJ}$.

An ongoing application of our work is to the field of AI ethics. This is based on: 1) interpreting our consequence relation as a form of *learning* hypotheses from data, 2) embedding the consequence relation in a broader *formal argumentation* framework, 3) including in the framework an account of ethical and epistemic concerns, as *attacks* that may inhibit learning.

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Predicate probability logic in a many-valued framework

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Probability logics aim to provide a formal framework for reasoning about uncertainty, enabling logical systems to handle statements that are probable rather than strictly true or false. Traditional approaches to probability logic are typically based on classical Boolean logic, extended by probabilistic modalities or operators. However, such frameworks often face limitations in expressive power and practical applicability. Some classical probability logics, e.g., [13], rely on threshold-based semantics, where statements are considered true only if their probability exceeds a certain value, which leads to undesirable logical properties such as the failure of adjunction. Other systems introduce numerous probabilistic modalities to capture different probability thresholds [8], constrain the expressive power of the logic [9], or use a complex axiomatization [5]. An alternative approach is to employ many-valued logics, particularly those based on the $[0,1]$-valued Łukasiewicz logic, where probability values are directly represented as the logic's truth degrees; arguably, this provides a more flexible and mathematically elegant treatment of probabilistic reasoning [12].

Łukasiewicz logic (propositional Ł and predicate Ł$\forall$, e.g., [11, 10]) is a well-known many-valued logic that generalizes classical Boolean logic by allowing truth values from the entire unit interval $[0,1]$ instead of just $\{0,1\}$. The logic has well-established algebraic semantics based on MV-algebras and enjoys (finite strong) completeness with respect to standard real-valued models as well as linear MV-algebras. The standard semantics interprets logical connectives and quantifiers by lattice ($\wedge$, $\vee$, $\forall$, $\exists$) and additive ($\otimes$, $\oplus$, $\rightarrow$, $\sim$) operations on $[0,1]$, which makes them suitable for expressing the additivity of probability.

Propositional fuzzy probability logic FP(Ł), introduced by Esteva, Godo, and Hájek [12], extends Łukasiewicz logic by a many-valued probability modality P. The syntax of FP(Ł) is two-layered, with (Boolean) non-modal formulae φ describing events and modal formulae (propositional combinations of modal atoms $\mathsf{P}\varphi$ by the connectives of Ł) expressing probability statements about the events. The modal atom $\mathsf{P}\varphi$ is informally interpreted as expressing the proposition "φ is probable", with the intended truth value in $[0,1]$ equal to the probability of the event φ. The standard semantics of FP(Ł) takes Boolean evaluations for possible worlds, sets of possible worlds as events, and interprets the modality P as a finitely additive probability measure over a set algebra of events. The axioms of FP(Ł) consist of those of propositional Łukasiewicz logic and the following three modal axiom schemata:

$$\mathsf{P}{\sim}\varphi \leftrightarrow {\sim}\mathsf{P}\varphi$$
$$\mathsf{P}(\varphi \rightarrow \psi) \rightarrow (\mathsf{P}\varphi \rightarrow \mathsf{P}\psi)$$
$$\mathsf{P}(\varphi \vee \psi) \leftrightarrow \big((\mathsf{P}\varphi \rightarrow \mathsf{P}(\varphi \wedge \psi)) \rightarrow \mathsf{P}\psi\big)$$

These axioms ensure the normality, monotonicity, and additivity of probability measure. The derivation rules of FP(Ł) are modus ponens and necessitation.

As shown in [1], FP(Ł) is as strong as the well known probability logic of Fagin, Halpern, and Megiddo [5]. Some even more expressive variants of FP(Ł) over various expansions of Łukasiewicz logic have been developed and studied by several authors [11, 7, 6, 2], allowing the representation of conditional or inconsistent probabilities of crisp or graded events. However, despite their expressiveness, the propositional nature of all these many-valued probability logics limits their capacity to express reasoning about quantified probabilistic statements and to represent more complex probabilistic concepts such as probability distributions over infinite domains. This limitation motivates the development of a predicate extension $\mathrm{FP}(\text{Ł}\forall)$ of $\mathrm{FP}(\text{Ł})$, which is the subject of the presented work.

The proposed predicate probability logic $\mathrm{FP}(\text{Ł}\forall)$ extends $\mathrm{FP}(\text{Ł})$ by integrating first-order logic into its two-layered framework, allowing formal reasoning about quantified probabilistic statements. In $\mathrm{FP}(\text{Ł}\forall)$, event formulae remain classical, while modal formulae take values from the real unit interval (or more generally a linear MV-algebra) and are governed by first-order Łukasiewicz logic Ł$\forall$. As in $\mathrm{FP}(\text{Ł})$, the many-valued (graded) modal atom $\mathsf{P}\varphi$ denotes the probability of the event φ. Additionally, $\mathrm{FP}(\text{Ł}\forall)$ includes (non-nestable, crisp, S5-style) alethic modalities $\Box$, $\Diamond$, and @ for reasoning about necessity, possibility, and truth in the actual world.

The models of $\mathrm{FP}(\text{Ł}\forall)$ consist of a collection W of first-order models for Łukasiewicz logic (possible worlds) that share a common domain D. As in the propositional case, the modality P is interpreted as a normalized many-valued measure on a set algebra (of events) over W, satisfying finite additivity (in the sense of $\oplus$ in Ł). In standard models that use $[0, 1]$ as the truth-value set, P is thus interpreted as a finitely additive probability measure, while nonstandard models allow for more general many-valued measures, such as hyperreal ones. The notions of evaluation of object variables in D, truth and validity in a model of $\mathrm{FP}(\text{Ł}\forall)$, and tautologicity and entailment are defined similarly as in first-order Łukasiewicz logic [11] (including the necessary nuances of safe models). The axiomatization and completeness proof of $\mathrm{FP}(\text{Ł}\forall)$ are carried out similarly to the propositional case [12, 11, 4], i.e., by showing that suitable translations between formal proofs in $\mathrm{FP}(\text{Ł}\forall)$ and Ł$\forall$ reduce the (finite strong) completeness of $\mathrm{FP}(\text{Ł}\forall)$ to that of Ł$\forall$.

Various logical laws can be shown to be valid in $\mathrm{FP}(\text{Ł}\forall)$. First, the probability operator P adheres to standard axioms of finitely additive (de Finettian) probability theory such as monotonicity and additivity. Furthermore, $\mathrm{FP}(\text{Ł}\forall)$ inherits all valid propositional laws for modal atoms from $\mathrm{FP}(\text{Ł})$. As expected, P is intermediate between $\Box$ and $\Diamond$ and incomparable with @. The metarule of necessitation applies to all four modalities: if a non-modal formula φ is a classical tautology, then $\Box\varphi$, $\Diamond\varphi$, $@\varphi$, and $\mathsf{P}\varphi$ are valid in $\mathrm{FP}(\text{Ł}\forall)$. The alethic modalities $\Box$, $\Diamond$, and @ behave classically and satisfy the usual laws of classical S5-style first-order modal logic (for non-nestable modalities), including the Barcan and converse Barcan formulae for $\Box$ and $\Diamond$. For probability, however, only the converse Barcan formulae are generally valid:

$$\models \mathsf{P}\forall x\varphi \to \forall x\mathsf{P}\varphi, \quad \models \exists x\mathsf{P}\varphi \to \mathsf{P}\exists x\varphi.$$

The expressive power of $\mathrm{FP}(\text{Ł}\forall)$ can be further enhanced in several directions. If needed, non-constant domains can be introduced by means of a crisp existence predicate, similarly as in dual-domain free logic. While predicate and function symbols are generally world-dependent in $\mathrm{FP}(\text{Ł}\forall)$, the rigidity of any of them can be enforced by axioms ensuring their values remain constant across all worlds, allowing the framework to model fixed structures (such as $\mathbb{N}$) and deterministic reasoning. Moreover, the underlying logic Ł$\forall$ can be replaced by its stronger expansions such as ŁΠ$\forall$ [3], whose multiplicative connectives enable modeling of conditional probabilities, Bayesian inference, expected value calculations, and representation (or approximation) of probability distributions. The syntax, semantics, and completeness proof of $\mathrm{FP}(\text{Ł}\forall)$ can be schematically generalized for two-layered many-valued probability logics over well-behaved expansions of Ł$\forall$. Furthermore, the approach extends beyond probability to other many-valued two-layered first-order modal logics (cf. [4] for the propositional case), making it applicable to doxastic, epistemic, and deontic reasoning.

The presentation will provide an overview of $\mathrm{FP}(\text{Ł}\forall)$, outlining its syntax, semantics, logical properties, and metamathematical results. Additionally, it will discuss the expressive power of $\mathrm{FP}(\text{Ł}\forall)$, outline its key extensions, and hint at their applicability to formalization of various aspects of probabilistic reasoning.

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Epistemic logic of crash failures[1]

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Simplicial complexes are a well-known semantic framework in combinatorial topology to model synchronous and asynchronous distributed systems. A common type of faults considered in synchronous computation is a *crash failure*, i.e., an agent ceasing to perform any actions, including responding to messages. In presence of crash failures, a live process may be uncertain whether another process has already crashed. In simplicial complexes, this is modeled semantically by considering so-called *impure* simplicial complexes. In this extended abstract, we discuss which object language is appropriate and expressive enough to reason about synchronous distributed systems with crash failures using the impure simplicial semantics.

Epistemic logic investigates knowledge and belief, and change of knowledge and belief, in multi-agent systems. Knowledge change was extensively modeled in temporal epistemic logics and in dynamic epistemic logics. Epistemic logical semantics is often based on *Kripke models*, which consist of an abstract domain of global states, or worlds, among which binary relations of accessibility (or indistinguishability, depending on the agents' epistemic strength) are defined, one for each agent.

Combinatorial topology [8] has been used in distributed computing to model concurrency and asynchrony since [4], including higher-dimensional topological properties [7, 10]. Geometric manipulations such as subdivision have natural combinatorial counterparts. *Simplicial models* consist of an abstract set of *vertices* representing agents' local states. These agent-colored vertices are combined into sets called *simplices*, with a standard chromatic restriction that each simplex contain no more than one vertex per agent. Global states of the system correspond to those simplices that are maximal with respect to set inclusion and are called *facets*. *Pure* simplicial complexes correspond to distributed systems without crashes, hence, require that each facet contain exactly one vertex for each of the agents. Crashed agents are modeled by allowing facets to have fewer vertices than the total number of agents, with the understanding that all agents missing from a facet are *dead*, i.e., have crashed, whereas all agents present in the facet (as a single vertex) are *alive*. The collection of sets of vertices (simplices) in a given simplicial model is assumed to be downward closed with respect to set inclusion, with the exception of the empty set. Proper subsets of any facet are called *faces* and can be viewed as partial global states of the system.

Fig. 1 provides examples of one pure ($\mathscr{C}_1$) and two impure ($\mathscr{C}_2$ and $\mathscr{C}_3$) simplicial models for a distributed system with three agents a, b, and c (see [3] for a formal definition):

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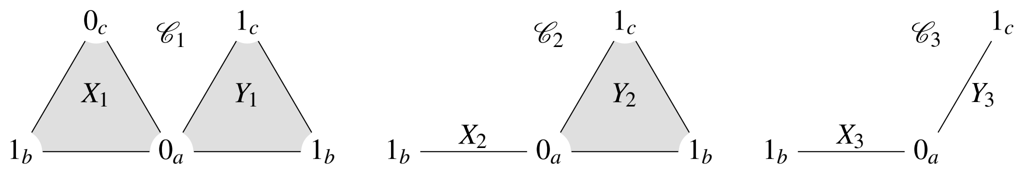

Figure 1: Impure and pure simplicial models

Each model $\mathscr{C}_i$ consists of two facets X_i and Y_i (global states) that agent a cannot distinguish, as evidenced by its vertex (local state) 0_a belonging to both. Model $\mathscr{C}_1$ is pure because its two facets (two gray triangles) X_1 and Y_1 have one vertex per agent each. Thus, a is sure that all agents are alive and knows the value of b's variable as it is true (depicted as 1_b) in both X_1 and Y_1. On the other hand, a does not know the truth value of c's variable as it is false (0_c) in X_1 and true (1_c) in Y_1. Models $\mathscr{C}_2$ and $\mathscr{C}_3$ are impure because each contains at least one facet with strictly less than three agents: agent c is dead in X_2 of $\mathscr{C}_2$ and in X_3 of $\mathscr{C}_3$. Note that, e.g., facet X_2 in the impure model $\mathscr{C}_2$ is an edge that can also be found in the pure model $\mathscr{C}_1$. However, there the corresponding edge is a side of a triangle, or in simplicial complex terms, is a face of a larger facet X_1, without itself being a facet. In both $\mathscr{C}_2$ and $\mathscr{C}_3$, agent a is unsure whether c is alive (and, additionally, whether b is alive in $\mathscr{C}_3$).

Note that our syntax assigns each propositional variable p_a, p_b, q_b to one of the agents, treating them as *local* in the sense of an agent's local state in distributed systems, which is always known by the agent. Thus, variable p_a pertaining to the local state of a should be known by a, as formalized by the *locality axiom* $K_a p_a \vee K_a \neg p_a$ where K_a represents agent a's knowledge [3, 5]. Local variables represent a natural but not the only choice. A logic of impure simplicial complexes with global variables can be found, e.g., in [6].

We believe that a proper logic for distributed systems should include both types of variables: local variables for describing agents' local states and global variables describing global properties of the system that need not be known to any agent. For instance, asynchronous systems are typically modeled to have global time that no agent has access to, making this global time a good example of a global variable that does not belong to any agent and is, generally, not known by any agent. Logically, the locality axiom should be applied to local variables only.

Another non-trivial question regards the effect agents' crashes have on the knowledge of live agents, in particular, on their knowledge of the local variables of crashed agents. Consider again $\mathscr{C}_2$ and $\mathscr{C}_3$ in Fig. 1. Does a know the value of, say, b's variable p_b there? The only obvious answer is that the value of p_b is known in $\mathscr{C}_2$ as it is true in both X_2 and Y_2. But what happens with p_b in facet Y_3 of model $\mathscr{C}_3$? And what does a know about it in facet X_3? Were p_b a global variable, as in [6], its truth value would have been determined by the whole facet Y_3, and the crash of agent b would not affect it. On the other hand, there is no universally acceptable way of assigning a truth value to a local variable p_b in facet Y_3. This prompted the introduction of the third truth value 'undefined' in [3]. Propositionally, this value is treated according to the 3-valued Weak Kleene Logic, with the undefined value "infecting" any propositional formula it participates in. The question about knowledge in presence of undefined values is more subtle. In global state X_3 of model $\mathscr{C}_3$, given that p_b is undefined in Y_3, (i) should a know p_b to be true based on X_3 alone, the sole facet where p_b is defined or (ii) should a not know p_b to be true because it is not true in Y_3, which a considers possible? Both options may seem reasonable at first but option (ii) has an undesirable consequence for the dual modality $\widehat{K}_a := \neg K_a \neg$, which stands for '$a$ considers it possible.' Indeed, if $\mathscr{C}_3, X_3 \nvDash K_a p_b$ according to (ii), then $\mathscr{C}_3, X_3 \vDash \widehat{K}_a \neg p_b$, i.e., agent a would have to consider it possible that p_b is false despite it not being false in any facet of $\mathscr{C}_3$. This consideration explains why option (i) was chosen in [3]. It should be noted that the resulting logic is different from the way modalities work in [2].

The resulting epistemic logic of impure simplicial complexes, based on the 3-valued Weak Kleene Logic on the propositional level and with local variables only, was axiomatized in [9]. As discovered in [1], the difficulty was that it did not satisfy the Hennessy–Milner property for the natural notion of bisimulation. Worse than that, no reasonable local definition of bisimulation relying on the standard back-and-forth relations would have Hennessy–Milner [9].

A failure of Hennessy–Milner often means that the language is not expressive enough. And the property lacking expressivity in terms of local variables only was quite obvious. Above, while we used the term "know", corresponding to the K_a modality for local variables, we resorted to "is sure that" regarding agents being alive or dead. The reason for this was that the latter was not expressible in the language with local variables only [1]. Hence, using "know" would have been misleading. Since one of the objectives in a distributed systems with crash failures is to reason in presence of crash failures, a language not expressive enough to talk about these crash failures in the object language is suboptimal.

Thus, both based on the desired applications and to solve the expressivity gap, we believe that, in addition to the local variables representing agents' local states, the object language for the logic of impure simplicial complexes should also include global variables, namely, atoms expressing that a particular agent is alive. In [1], it was shown that the logic with such atoms a for each agent a does indeed possess the Hennessy–Milner property. We are currently preparing for submission a manuscript with a complete axiom system for this logic, which extends that from [9] for local variables only.

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Connecting state-based and team-based inquisitive logic

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Introduction

We develop entailment-preserving translations between the inquisitive logics InqBQ ([1],[2]) and InqBT ([1],[3]). InqBQ is the standard system of first-order inquisitive logic, interpreted in a state-based semantics; InqBT (also known as *weak intuitionistic dependence logic*) interprets the same language using team semantics ([4],[5]). We use the translations to connect key metatheoretic open problems of the two systems (entailment-compactness and the recursive enumerability of their validities) and to transfer results from fragments of InqBQ to InqBT.

Inquisitive logic is a conservative extension of classical logic that provides frameworks where statements and questions can be analyzed uniformly. In InqBQ, the semantics is based on *information states*, sets of worlds which model uncertainty about the state of affairs; the language can express questions about the state of affairs, such as "*what is the extension of P?*" where P is a unary predicate. In InqBT, the semantics is based on *teams*, sets of assignments which can be seen as modeling uncertainty about the values of variables; the language allows us to formalize questions such as "*what is the value of x?*", as well as dependencies between variables, including the dependence atoms $=(x_1, \ldots, x_n; y)$ used in Dependence Logic ([6]).

Developing translations between InqBQ and InqBT deepens our understanding of the connection between these logics. The equivalence of open problems facilitates the attainment of future results, while the transfer of the fragments' properties sheds light on the metatheoretical properties of InqBT. The translations themselves also turn out to be intrinsically interesting: the main ideas behind their definition are to some extent independent of the specific systems in question, suggesting a possible generalization of the present methods to other systems.

Technical background

The language of InqBQ and InqBT extends the language of first-order classical logic with two inquisitive operators, $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and $\exists$. Formally, it is given by the following syntax, where p ranges over first-order atom in a signature Σ: $\varphi ::= \bot \mid p \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \varphi \mid \forall x\varphi \mid \exists x\varphi$. Formulas without $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ or $\exists$ are referred to as *classical* formulas, and can be identified with the formulas of standard first-order logic, where $\neg, \vee, \exists$, are defined in standard ways. For simplicity, here we restrict attention to *relational* signatures, which include predicates but no function symbols, and do not consider the identity predicate; however, the results generalize. The two logics diverge in their semantics. InqBQ is interpreted in a state-based semantics. Models for InqBQ are tuples $M = \langle W, D, I\rangle$ where W is the set of possible worlds of M, D is the domain of quantification, and I assigns to each $w \in W$ a classical interpretation function I_w into D: thus, each $w \in W$ is associated with a standard relational structure, $\langle D, I_w\rangle$. Formulas φ are interpreted in terms of a relation of *support* ($M, s \vDash_g \varphi$) relative to an *information state* s, defined as a subset of W, and an assignment g into D. For classical formulas, support boils down to (classical) truth at each possible world in the state. Inquisitive disjunctions $\varphi \vee\!\!\!\vee \psi$ are supported by a state when one of the disjuncts is supported. Inquisitive existentials $\exists x\varphi(x)$ are supported when $\varphi(d)$ is supported for some $d \in D$. Entailment is defined as preservation of support: $\Phi \vDash_{\mathsf{InqBQ}} \psi$ iff $\forall M, s, g : M, s \vDash_g \Phi \implies M, s \vDash_g \psi$.

InqBT is interpreted using team semantics ([4],[5]). Formulas are evaluated in terms of support relative to a standard relational structure $\mathcal{M} = \langle \mathcal{D}, \mathcal{I} \rangle$ and a *team* T, defined as a set of assignment functions into $\mathcal{D}$. The support relation of InqBT (written as $\mathcal{M} \vDash_T \varphi$) is defined exactly like that of InqBQ, but with teams in lieu of states. In particular, unlike in dependence logic, the quantifiers $\forall x/\exists x$ are interpreted by setting the value of x constantly throughout the team. The dependence atom $=\!(x_1, \ldots, x_n; y)$ of [6] can be defined in this logic as $\lambda x_1 \wedge \cdots \wedge \lambda x_n \to \lambda y$, where $\lambda x_i := \exists z(x_i = z)$. Entailment ($\Phi \vDash_{\mathsf{InqBT}} \psi$) is defined as preservation of support for arbitrary $\mathcal{M}$ and T. An obvious parallel can be drawn between the two systems: both rely on **sets** of classical evaluation points, but while InqBQ uses a set of possible worlds (associated with a set of relational structures) and a single assignment, InqBT uses a single relational structure and a set of assignments. Modulo this difference, the semantics is exactly the same. This suggests the existence of a connection between these logics, motivating the present work. Research around InqBQ is significantly more developed than that around InqBT. Nonetheless, major metatheoretical questions remain open for both systems. In particular, it is not known whether these logics are recursively enumerable and whether they are entailment-compact (in the sense that, for all Φ and ψ, $\Phi \vDash \psi$ implies $\Phi_0 \vDash \psi$ for some finite $\Phi_0 \subseteq \Phi$). These questions have been answered in the positive for two broad syntactical fragments of InqBQ, namely Clant (where only classical formulas are allowed as implication antecedents) and Rex (where $\exists$ can only appear in the antecedents of implications), for which complete proof systems have also been developed ([7],[8]).

Defining the translations

Observing the semantics of InqBQ and InqBT, the natural approach is to replicate the behavior of states with teams and vice-versa. To do so, we use variables and constants to encode, respectively, the role of possible worlds and that of assignments.

InqBQ to InqBT For the InqBQ to InqBT direction, we define a translation $\sharp$ as follows.

• **Models** We translate a model $M = \langle W, D, I \rangle$ with an InqBT model $\mathcal{M}^\sharp = \langle W \uplus D, \mathcal{I}^\sharp \rangle$. To distinguish worlds and individuals, we introduce a predicate Ind, verified only by the elements of D. In InqBQ, predicate interpretation changes with worlds. To make predicates sensitive to changing assignments, we translate every n-ary R into the $n+1$-ary R' and introduce a variable that simulates the role of possible worlds in the interpretation of R. An $n+1$-tuple of $(W \uplus D)^{n+1}$ satisfies R' when the first element is a world w and the rest are individuals that satisfied R in w. Formally: $(w, \overline{d}) \in \mathcal{I}^\sharp(R') \iff w \in W, \overline{d} \in D^n,$ and $\overline{d} \in I_w(R)$.

• **States** The role of InqBQ states is taken over by teams in InqBT. Thus, we associate each world w and assignment g to an assignment g_w by letting $g_w(x_0) = w$ and $g_w(x_{n+1}) = g(x_n)$. Given a state s and assignment g, we define the corresponding team as $T^g_s = \{g_w \mid w \in s\}$.

• **Formulas** We define the translation $\varphi^\sharp$ of a formula φ by induction. For atoms, we introduce x_0 and shift all other variables. We restrict the range of quantifiers to D using Ind.

- $(R(x_{i_1}, \ldots, x_{i_n}))^\sharp := R'(x_0, x_{i_1+1}, \ldots, x_{i_n+1})$
- $(\varphi \circ \psi)^\sharp := \varphi^\sharp \circ \psi^\sharp$, for $\circ \in \{\wedge, \to, \vee\!\!\!\vee\}$
- $(\forall x_i\varphi)^\sharp := \forall x_{i+1}(\text{Ind}(x_{i+1}) \to \varphi^\sharp)$ and $(\exists x_i\varphi)^\sharp := \exists x_{i+1}(\text{Ind}(x_{i+1}) \to \varphi^\sharp)$

The translation $\sharp$ satisfies two crucial properties:

1. **Preservation of support**: for any formula φ, model M, state s and assignment g,
$$M, s \vDash_g \varphi \iff \mathcal{M}^\sharp \vDash_{T_s^g} \varphi^\sharp.$$
2. **Characterization**: the set of model-team pairs that arise as $\sharp$-images of state-assignment pairs is characterized in InqBT by a set of formulas, called Γ.

Together, **1.** and **2.** imply that $\sharp$ preserves entailment: for any set of formulas $\Phi \cup \{\psi\}$,
$$\Phi \vDash_{\mathsf{InqBQ}} \psi \iff \Gamma, \Phi^\sharp \vDash_{\mathsf{InqBT}} \psi^\sharp.$$

InqBT to InqBQ For the InqBT to InqBQ direction, we define a translation $\flat$.

• **Models and teams** In InqBT teams, variables are evaluated differently by different assignments. To emulate this behavior with possible worlds, we replace each variable x_i with a corresponding constant a_i. Given a team, we introduce for each assignment a world that interprets a_i like the assignment evaluates x_i. Formally, for a model $\mathcal{M} = \langle \mathcal{D}, \mathcal{I} \rangle$ and a team T, we define $M_T^\flat = \langle W^\flat, \mathcal{D}, I^\flat \rangle$, where $W^\flat = \{w_g \mid g \in T\}$, $I_{w_g}^\flat(R) = \mathcal{I}(R)$ and $I_{w_g}^\flat(a_i) = g(x_i)$.
• **Formulas** For a formula φ, we define $\varphi^\flat$ inductively. We replace free occurrences of x_i with a_i, but ensure that *bound* occurrences of variables are not replaced in quantified formulas.

- $(R(x_{i_1}, \dots, x_{i_n}))^\flat := R(a_i, \dots, a_n)$
- $(\varphi \circ \psi)^\flat := \varphi^\flat \circ \psi^\flat$, for $\circ \in \{\wedge, \rightarrow, \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}\}$
- $(\forall x_i \varphi)^\flat := \forall x_i(\varphi^\flat[x_i/a_i])$ and $(\exists x_i \varphi)^\flat := \exists x_i(\varphi^\flat[x_i/a_i])$

The translation $\flat$ also verifies properties analogous to **1.** and **2.** and preserves entailment.

Repercussions

Equivalence of open problems The translations, being entailment-preserving, allow for simple proofs of equivalence for two major open problems of InqBQ and InqBT:

InqBQ is entailment-compact $\iff$ InqBT is entailment-compact

InqBQ validities are recursively enumerable $\iff$ InqBT validities are recursively enumerable

Transferring properties of fragments Similarly, we can transfer positive solutions to these open problems from the Rex and Clant fragments of InqBQ to their InqBT counterparts:

- The Rex and Clant fragments of InqBT are entailment-compact.
- The sets of Rex validities and Clant validities in InqBT are recursively enumerable.

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RNmatrices for modal logics

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We present a novel approach for constructing semantics for both: normal and non-normal modal logics, based on *restricted* non-deterministic semantics. This proves to be very versatile in the sense that given a finite axiomatic characterization of a modal system, one can construct a semantics, such that the given axiomatic system is sound and complete.

We begin our study with the weakest system of modal logic **M**. This system is an expansion of classical propositional logic with an unary operator $\ominus$ and is characterized as follows (note that no interpretation for $\ominus$ is given):

- **M** contains all (classical) tautologies
- **M** is closed under uniform substitution
- **M** is closed under Modus Ponens

For the presentation, we consider a propositional modal signature Σ with unary connectives $\neg$ and $\ominus$ (classical negation and modality, respectively) and a binary connective $\to$ (material implication). Let $\mathcal{V}$ be a denumerable set of propositional variables $\mathcal{V} = \{p_0, p_1, \ldots\}$ and let $For(\Sigma)$ be the algebra of formulas over Σ freely generated by $\mathcal{V}$.

The formulas of **M** (and its extensions) are interpreted by a set of four-valued non-deterministic matrices (Nmatrices) defined from swap structures (see for instance (Carnielli and Coniglio, 2016, Ch. 6)) in which each truth-value is an ordered pair (or *snapshot*) $z = (z_1, z_2)$ in $\mathbf{2}^2$, for $\mathbf{2} = \{0, 1\}$. Here, z_1 and z_2 represent, respectively, the truth value of A and of $\ominus A$ for a given formula A over Σ. This produces four truth-values: $(1,0)$, $(1,1)$, $(0,1)$, and $(0,0)$, which, depending on the interpretation of $\ominus$, could be interpreted as true but not necessary, necessarily true, false but necessary, and false and not necessary. Let V_4 be the set of such truth-values. Accordingly, the set of designated values will be $D_4 = \{z \in V_4 \ : \ z_1 = 1\} = \{(1,0), (1,1)\}$.

Because of the intended meaning of the snapshots, negation and implication between snapshots are computed over $\mathbf{2}$ in the first coordinate, while the second one can take an arbitrary value, which will be denoted by $*$. Furthermore, the interpretation of $\ominus$ is a multioperator which simply 'reads' the second coordinate, while the second coordinate (in that case corresponding to $\ominus \ominus A$) will be arbitrary at this point. Let $\sim$ and $\Rightarrow$ denote the Boolean negation and the implication in $\mathbf{2}$, then:

$$\begin{aligned} \tilde{\neg}\, z &:= (\sim z_1, *); \\ z \,\tilde{\to}\, w &:= (z_1 \Rightarrow w_1, *) \\ \tilde{\ominus}\, z &:= (z_2, *). \end{aligned}$$

Let $\mathcal{M} = \langle V_4, D_4, \mathcal{O}\rangle$ be the obtained 4-valued Nmatrix, where $\mathcal{O}(\#) = \tilde{\#}$ for every connective $\#$ in Σ. Now, let $\mathcal{F}$ be the set of all the valuations over the Nmatrix $\mathcal{M}$, such that $v \in \mathcal{F}$ iff $v : For(\Sigma) \to V_4$ is a function satisfying for every connective $\#$ in Σ the following property:

$$v(\#(A_1, \ldots, A_n)) \in \tilde{\#}v(A_1), \ldots, v(A_n)$$

We will now write any valuation $v \in \mathcal{F}$ over the Nmatrix $\mathcal{M}$, $v = (v_1, v_2)$ such that $v_1, v_2 : For(\Sigma) \to \mathbf{2}$. Hence, $v(A) = (v_1(A), v_2(A))$ for every formula A. This means that, for all formulas A and B:

- $v(A) \in D_4$ iff $v_1(A) = 1$;
- $v_1(\neg A) = {\sim} v_1(A)$;
- $v_1(\ominus A) = v_2(A)$;
- $v_1(A \to B) = v_1(A) \Rightarrow v_1(B)$.

The logic **M** generated by the Nmatrix $\mathcal{M}$ is then defined as follows: $\Gamma \vDash_{\mathbf{M}} A$ iff, for every $v \in \mathcal{F}$: if $v_1(B) = 1$ for every $B \in \Gamma$ then $v_1(A) = 1$.

Let $\mathcal{H}$ be the standard Hilbert calculus for classical propositional logic **CPL**, presented in the signature Σ (that is, no axioms nor rules for $\ominus$ are given, *Modus Ponens* being the only inference rule). It is easy to prove the following result:

Theorem (Soundness and completeness of $\mathcal{H}$ w.r.t. $\mathcal{M}$): For every $\Gamma \cup \{A\} \subseteq For(\Sigma)$ it holds: $\Gamma \vdash_{\mathcal{H}} A$ iff $\Gamma \vDash_{\mathbf{M}} A$.

We will now introduce and exploit what in Coniglio and Toledo (2021) was called *restricted Nmatrices* (RNmatrices). RNmatrices have the form $\mathcal{RM} = \langle \mathcal{M}, \mathcal{F}'\rangle$, where $\mathcal{M}$ is the Nmatrix for **M** and $\mathcal{F}' \subseteq \mathcal{F}$ and the set $\mathcal{F}'$ is closed under substitutions[1] — so each RNmatrix will be *structural*. The aim of the restriction is to satisfy certain modal axiom(s) and, later on, modal rules. As proved in Coniglio and Toledo (2021), any structural RNmatrix generates a Tarskian and structural consequence relation defined as expected: $\Gamma \vDash_{\mathcal{RM}} A$ iff, for every $v \in \mathcal{F}'$: if $v(B) = 1$ for every $B \in \Gamma$ then $v(A) = 1$.

For the purpose of illustration in this abstract, we give three modal axioms and the restrictions imposed by these axioms on valuations as an example:

Axiom	Valuations
$\ominus A \to A$	$v_2(A) \leq v_1(A)$
$\ominus A \to \ominus \ominus A$	$v_2(A) \leq v_2(\ominus A)$
$\ominus(\ominus A \to A) \to \ominus A$	$v_2(\ominus A \to A) \leq v_2(A)$

Let $\mathcal{H}_{\mathsf{Ax}}$ be the extension of $\mathcal{H}$ by some modal axioms and $\mathcal{RM}_{\mathsf{Ax}}$ the corresponding RNmatrix. We then can prove the following soundness and completeness result:

Theorem (Soundness and completeness of $\mathcal{H}_{\mathsf{Ax}}$ w.r.t. the RNmatrix $\mathcal{RM}_{\mathsf{Ax}}$) Let $\Gamma \cup \{A\} \subseteq For(\Sigma)$. Then: $\Gamma \vdash_{\mathcal{H}_{\mathsf{Ax}}} A$ iff $\Gamma \vDash_{\mathcal{RM}_{\mathsf{Ax}}} A$.

[1] A *substitution* over the signature Σ of $\mathbf{L}^R$ is a function $\sigma : \mathcal{V} \to For(\Sigma)$. Since $For(\Sigma)$ is an absolutely free algebra, each σ can be extended to a unique endomorphism in $For(\Sigma)$ (which will be also denoted by σ). That is, $\sigma : For(\Sigma) \to For(\Sigma)$ is such that $\sigma(\#A) = \#\sigma(A)$ for $\# \in \{\neg, \ominus\}$, and $\sigma(A \to B) = \sigma(A) \to \sigma(B)$. The set of substitutions over σ (seen as endomorphisms in $For(\Sigma)$) will be denoted by $Subs(\Sigma)$.

Let us now consider extensions $\mathbf{L}$ of $\mathbf{M}$ characterized by a Hilbert calculus $\mathcal{H}_{\mathsf{Ax}}$. Let $\mathcal{H}^R_{\mathsf{Ax}}$ be the Hilbert calculus obtained from $\mathcal{H}_{\mathsf{Ax}}$ by adding the global inference rule (by a global inference rule we mean a rule, where the premises and conclusion are theorems):

$$\frac{E_1 \;\; E_2 \;\; \dots \;\; E_s}{E} \quad (\mathbf{R})$$

The logic characterized by $\mathcal{H}^R_{\mathsf{Ax}}$, will be denoted by $\mathbf{L}^R$. The semantic counterpart of $\mathcal{H}^R_{\mathsf{Ax}}$ will be constructed by recursively restricting the set of all valuations:

Definition ($\mathbf{R}$-Level valuations): We define the set $\mathcal{F}^R_{\mathsf{Ax}}$ inductively as follows:

- $\mathcal{F}^0_{\mathsf{Ax}} = \mathcal{F}_{\mathsf{Ax}}$
- $\mathcal{F}^{m+1}_{\mathsf{Ax}} = \big\{v \in \mathcal{F}^m_{\mathsf{Ax}} \,:\, \forall\, A \in For(\Sigma), \forall\, \sigma \in Subs(\Sigma)$, if $A = \sigma(E)$ and $\forall\, 1 \leq i \leq s, \forall w \in \mathcal{F}^m_{\mathsf{Ax}}(w_1(\sigma(E_i)) = 1)$, then $v_1(A) = 1\big\}$
- $\mathcal{F}^R_{\mathsf{Ax}} = \bigcap_{m=0}^{\infty} \mathcal{F}^m_{\mathsf{Ax}}$

Definition:
(1) A is valid in $\mathbf{L}^R$, denoted by $\models_{\mathbf{L}^R} A$, if $v_1(A) = 1$ for every $v \in \mathcal{F}^R_{\mathsf{Ax}}$.
(2) A is a semantical consequence of Γ in $\mathbf{L}^R$, denoted by $\Gamma \models_{\mathbf{L}^R} A$, if either A is valid in $\mathbf{L}^R$, or $B_1 \to (B_2 \to (\dots \to (B_k \to A)\dots)$ is valid in $\mathbf{L}^R$ for some nonempty finite set $\{B_1, \dots, B_k\} \subseteq \Gamma$.

We can then prove the following theorem.

Theorem (Soundness and Completeness of $\mathcal{H}^R_{\mathsf{Ax}}$ w.r.t. level valuation semantics): If $\Gamma \vdash_{\mathcal{H}^R_{\mathsf{Ax}}} A$ then $\Gamma \models_{\mathbf{L}^R} A$.

Based on this, the aim of this presentation is then as follows: we will introduce RNmatrices for the minimal modal logic, discuss extensions, without any rules for the modal operator, of $\mathbf{M}$ by making the appropriate restrictions to the set of all valuations. General soundness and completeness results will be presented for all the extensions. This will be followed by a discussion how the semantics can be enriched with global modal rules, and thus providing a new semantics for normal and non-normal modal logics.

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On the conservativity of the minimalist foundation over arithmetic

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The notion of conservativity traces back to the method of ideal elements, which has been widely used in various areas of mathematics throughout history. Reflecting on this method, Hilbert made conservativity a crucial ingredient of his foundational program. One of Hilbert's aims was to show the conservativity of infinitary (ideal) mathematics over finitary (real) mathematics, thereby justifying the use of infinitary methods.

Given two theories T' and T, where T' is an extension of T, we say that T' is *conservative* over T, if every sentence φ which is expressible in the language of T and provable in T', is already provable in T. Conservativity results have a pleasant philosophical moral: if we extend a certain theory by new principles that are proven to be conservative, then such an extension can be considered 'harmless'. Therefore, we are free to adopt these new principles in our reasoning without the need for further justification. This was the main philosophical insight behind Hilbert's Program.

After Hilbert, conservativity became a key topic in logic and foundations of mathematics, promoting the design of foundational systems, which, on the one hand, have rich and expressive languages and allow for a smooth development of mathematics, and, on the other hand, can be reduced to more basic and fundamental theories such as those of arithmetic. In particular, the Reverse Mathematics program (Simpson , 2009) has shown that non-trivial mathematical results can be already derived within theories that are conservative over arithmetic (and, in some cases, even over primitive recursive arithmetic).

The focus of this work is on the Minimalist Foundation, a two-level intuitionistic dependent type theory, first conceived in (Maietti, Sambin, 2005) and later fully formalized in (Maietti, 2009), proposed as a common core among the plurality of foundations: definitions, theorems and proofs written in its formalism can be mechanically translated into the language of the most relevant foundations for mathematics. More precisely, the whole system, called **MF**, consists of an intensional type theory, **mTT**, an extensional type theory, **emTT**, and an interpretation of the latter within the former through a setoid model construction (Maietti, 2009). The intensional level **mTT** serves as a functional programming language, enjoys good computational properties, and admits a Kleene-style realizability interpretation, allowing in particular for the extraction of programs from proofs. The extensional level **emTT** is intended to be the actual theory in which mathematics is developed, featuring principles (e.g. function extensionality, quotients) that are particularly convenient for mathematical practice.

Intuitively, each of the two levels extends predicate logic with its own type system that, in particular, can interpret the language of arithmetic. Although the two type systems behave differently, especially in the treatment of equality in higher types, as far as arithmetic is concerned it has been proven in (Sabelli , 2024) that no difference arises: the theories $\mathbf{emTT}$ and $\mathbf{mTT}$ derive the same formulas expressible in the language of arithmetic. This fact allows us to do uniform investigations and statements about the strength of the Minimalist Foundation relative to arithmetic which are independent of the choice of a particular level.

Both levels of $\mathbf{MF}$ feature a distinction among their types, which can be either *sets* or *collections*. Such distinction is akin to that between sets and classes in axiomatic set theory; however, in $\mathbf{MF}$, the distinction is employed with a peculiar qualitative stance, reflecting a more philosophical distinction between inductively generated and open-ended domains. Intuitively, a set is an inductively generated domain, such as the natural numbers $\mathbb{N}$, whose canonical elements are fixed in advance and remain unaffected by possible extensions of the ambient theory. On the contrary, the elements of a collection such as the subsets of natural numbers $\mathcal{P}(\mathbb{N})$ could be potentially ever undetermined and may increase in number as soon as the theory becomes more expressive. In particular, the presence of collections allows higher-order reasoning inside $\mathbf{MF}$. We will be especially interested in the fragment of $\mathbf{MF}$ having only set constructors, called the *first-order fragment* of $\mathbf{MF}$ and denoted as $\mathbf{MF}_{set}$.

The starting point of our work is the well-known theorem by Beeson stating the conservativity of the first-order fragment (i.e. the fragment without universes) of Martin-Löf type theory over Heyting Arithmetic (Beeson , 1985). Firstly, we show how this result carries over to $\mathbf{MF}_{set}$ too. Then, our main result concerns the *classical version* $\mathbf{MF}^c_{set}$ of $\mathbf{MF}_{set}$ obtained by turning the default intuitionistic logic into classical logic with the addition of the Law of Excluded Middle.

$$\frac{\varphi \; prop}{\varphi \vee \neg\varphi \; true}$$

Using the equiconsistency of $\mathbf{MF}$ with its classical version $\mathbf{MF}^c$ established in (Maietti, Sabelli, 2025), we show that $\mathbf{MF}^c_{set}$ is conservative over Peano Arithmetic. This modularity with respect to the addition of classical logic is a peculiarity of $\mathbf{MF}$ and is not achievable in the other major constructive foundations since, as explained in (Contente, Maietti, 2024), there the addition of classical logic results in significantly stronger systems.

Then, we conjecture how such conservativity result could be extended in two other directions by considering the addition of a computability axiom, or a second-order language, as explained more precisely in the following.

Formal Church Thesis. Recall that, within a sufficiently expressive theory T, the *formal Church Thesis* (not to be confused with the Church-Turing thesis) is a statement of T asserting that all its number-theoretic functions are computable. More formally, it states that for every number-theoretic function f, there exists a Turing machine with Gödel code e such that on each argument x, the output $\mathsf{U}(y)$ extracted from the computation history y of the machine e on input x is equal to the value $f(x)$. Such encoding is possible through Kleene's T predicate. Thus, the formal Church Thesis is formalized as follows.

$$\forall f \in \mathbb{N}^{\mathbb{N}}\, \exists e \in \mathbb{N}\, \forall x \in \mathbb{N}\, \exists y \in \mathbb{N}\,.\, \mathsf{T}(e,x,y) \wedge \mathsf{U}(y) = f(x)$$

In $\mathbf{MF}_{set}$, the formal Church Thesis does not hold; however, we conjecture that its assumption does not increase the set of provable arithmetical statements of $\mathbf{MF}_{set}$.

Second-order Language A classical result of Reverse mathematics is the conservativity of the *Arithmetical Comprehension Axiom* subsystem $\mathbf{ACA}_0$ of second-order arithmetic over (first-order) Peano Arithmetic. The key restriction of $\mathbf{ACA}_0$ is that the induction principle of natural numbers works only for *arithmetical* formulas, that is formulas in which quantification is restricted to natural numbers.

We discuss the possibility of adapting this result to the Minimalist Foundation by showing that its whole system including both sets and collections is still conservative over Peano Arithmetic as soon as an analogous restriction on the induction strength of the natural numbers is imposed.

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Impredicativity and schematic generality[1]

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A definition is said to be (syntactically) impredicative if it contains a quantifier that binds a variable of the same order of the *definiendum*. A definition is said to be (semantically) impredicative if it quantifies over a totality to which the *definiendum* belongs. A long-standing philosophical debate has focused on impredicativity as the source of vicious circularity, which in turn is responsible for inconsistency (Russell, 1908), (Poincare, 1906), of semantic instability (Poincare, 1906), definitional failure (Russell, 1908), (Dummett, 1991) and, more recently, of a violation of the potential nature of the infinite domains usually involved in such definitions (LinneboShapiro, 2023).

My preliminary aim is to disentangle impredicativity and vicious circularity, with a corresponding distinction between two – often overlapping (Dummett, 1991), (LinneboShapiro, 2023) – notions of predicativism, obtained respectively by a restriction of the quantification domain and by a different interpretation of the quantification itself. I will call only the first strategy predicativist, since it concerns (the notion of totality involved in) the above definition of impredicativity. On the contrary, I will focus on different implementations of the second strategy, which we can consider as alternative versions of non-viciously circular impredicativity. I will then examine the phenomenon of impredicativity in the light of different accounts of quantification, in order to test the hypothesis that the vicious circularity usually attributed to it arises only in virtue of the meaning of quantification in classical logic, as the exhaustive (possibly infinite) classical conjunction or disjunction of its instances, and of the corresponding notion of generality involved. In this framework, circularity arises because not only the *definiendum* of an impredicative definition is one of these instances – namely one of the candidate substitution of the variables bound by the quantifier and its denotation one of its possible values – but because the classical meaning of the quantification requires an exhaustive examination of all of these instances.

Three alternative approaches to impredicativity, based on different interpretations of the quantification, will be explored and compared. These are inspired respectively by original insights of Weyl, Carnap and Russell, and have been recently rediscovered, in the light of current logical developments. Such approaches share an explanation of the generality involved in the quantification that is currently defined *generic*, i.e. not instance-based (Linnebo, 2022), and allow to save syntactically impredicative definititions from the ban included in the traditional reading of Russell's Vicious Circle Principle – VCP, (Russell, 1908).[2] More precisely, they allow the VCP itself – and especially the notion of totality involved – to be relativised with respect to the classical meaning of quantification. Despite their similar effects on the phenomenon of the impredicativity, different motivations will be identified, justifying different non-classical formalisations and ultimately revealing a different notion of (even generic) generality.

The first non-classical treatment of impredicativity was proposed by Weyl (Weyl, 1921) and is based on the adoption of intuitionistic logic, in which the truth of universal statements

[1]The presentation of this work is founded by the program "Internationale Kommunikation" of the ÖFG – Österreichische Forschungsgemeinschaft.

[2]"No totality can contain members defined in terms of itself".

does not depend on the verification of their instances – impossible in the case of infinite domains – but "lies in the essence" shared by all of them. A recent implementation of this approach has been formalised in semi-intuitionistic logic (Linnebo, 2022), (LinneboShapiro, 2023).[3] Also in this case, the universal quantification expresses something stronger than the absence of counterexamples (in that it is not dual to the existential quantifier), relying instead on fully general facts about the properties involved in the generalisation. Such an approach is particularly useful in the case of a potentialist framework, because it makes the universal generalisation available from the beginning of the generative process of the instances, and independently of the stages. The non-duality of quantifiers is essential for the generic explanation of the truth of universal generalisation, but on the other hand it suggests a doubt about the generality of existential quantification, the truth of which depends on the exhibition of a witness.

A competing approach follows a Carnapian insight (Carnap, 1931) and justifies impredicativity on the basis of what he called the "specific generality" (as opposed to "numerical generality") of the quantification involved, whose behaviour is independent of running through all the individual cases, but relies on the uniformity of the proofs of the corresponding universal statements. The generality pointed out by Carnap anticipated what is currently called "schematic generality" (CrosillaLinnebo, 2023) and is usually attributed to parameters. This approach suggests a constructivist program and supports the impredicative developments of type theory in the polymorphic lambda calculus (Pistone, 2018), (FruchartLongo, 1999). In this framework, impredicativity is allowed in form of parametric polymorphism, i.e. terms/programs that work with inputs of different types (including, possibly, its own type) to give terms as outputs – and then can have infinitely many types. In this setting, impredicativity does not introduce vicious circularity in virtue of the regularity of polymorphic terms. As recently proved (FruchartLongo, 1999), the behaviour of a polymorphic (i.e. impredicative) term on any generic input type implies its uniform behaviour on all the input types,[4] thus guaranteeing a generalisation by means of a *prototype proof* (Goldfarb, 1987) of a generic type rather than the collections of all the individual proofs.

The last account I introduce in the debate and explore comes from Russell's (Russell, 1903) - long unheard - proposal to distinguish the universal propositions introduced by "all" from those introduced by "any" (and, as recently proposed, the existential propositions introduced by "some" from those introduced by "a" – cf. (Zardini, 2015)). Substructural insights into the ambiguity of quantifiers allow us to distinguish additive from multiplicative meanings of quantification. While the universal multiplicative quantifier requires an evaluation on all the possible instances, the additive counterpart is based on the evidence of any (then a singular and generic) instance.[5] These two kinds of quantifiers inherit the properties of the corresponding multiplicative and additive conjunction and disjunction (Paoli, 2005), (Zardini, 2015). In particular, the non-contractive substructural approach allows - by renouncing the metarule of adjunction - to distinguish two forms of universal generalisation and to formalise the distinction between anything and everything. For these reasons, they seem useful for disentangling the different meanings of quantification and the corresponding effects of impredicativity. In particular, I will show that, in the case of impredicative definitions, vicious circularity presup-

[3]Semi-intuitionistic logic can be formalised by adding to plural intuitionistic logic the Bounded Omniscence ($\forall yy((\forall x \prec yy)(\phi(y) \vee \neg\phi(y)) \rightarrow (\forall x \prec yy)\phi(y) \vee (\exists x \prec yy)\neg\phi(y))$) and the Law of Excluded Middle for atomic formulas ($\forall \overline{x}(P\overline{x} \vee \neg P\overline{x})$), in order to obtain that certain restricted generalizations behave classically.

[4]Cf. (FruchartLongo, 1999) The Genericity Theorem: Let M and N have type $\forall X.\sigma$. Then: $(\exists \tau, M\tau =_{Fc} N\tau) \rightarrow M =_{Fc} N$.

[5]Correspondingly, while the multiplicative existential quantification is verified on the basis of a connection between all the possible instances, the additive counterpart requires a single witness.

poses a multiplicative reading of quantifiers and the corresponding instance-based explanation of the generality involved. On the contrary, the additive reading of quantifiers makes the same impredicative definitions harmless by formalising a new kind of schematic generality.

As noted above, all the approaches share the idea that the account of predicativism depends strictly on the notion of generality involved in the quantification (on which the impredicativity depends) and in the totality mentioned in the VCP. However, on the semi-intuitonistic route, we still presuppose a notion of *proper* totality, namely the kind of generality inherent in the infinite nature of every infinite collections (e.g. integers, real numbers...); on the other hand, on the type-theoretic and substructural routes, we presuppose what we can call *schematic* totality, namely the generality of the syntactic rules governing substitution, instantiation and elimination processes.

Finally, the recent thesis of predicativism as a form of potentialism (LinneboShapiro, 2023) will be discussed, particularly in the light of the last two accounts of impredicativity presented above. The strategy followed so far is compatible with the potentialist thesis, but, by prioritising the choice of the logic respect to the analysis of the domain of quantification, it aims to provide an analysis that could in principle be neutral with respect to this thesis.

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Conceptually grounded quantification

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This paper lifts a logic for contextually relativized quantification to an intensional format. The result is a method of conceptually grounded quantification, or quantification through conceptual windows, as it is also called here. The method is shown to formally and methodologically improve upon currently popular approaches to quantification in a modal setting.

1. Contextually Relativized Quantification

This paper takes a start from a very simple logic (proof- and model-theory) for contextually relativized quantification. Contextually relativized quantifiers are dressed with indices that serve to identify and distinguish the various domains, possibly distinct, that the quantifiers are taken to range over. The proof rules are given in Fitch-style natural deduction format and they are all standard except for the fact that (i) existentially generalization over a certain domain is licensed only for variables that have been declared in that domain, and that (ii) existential instantiation employs reasoning about arbitrary variables that are declared in that domain.

∃-Introduction (I∃)

$$\begin{array}{lll} & \vdots & \\ \text{m.} & [z/x]\phi & \\ & \vdots & \\ \text{n.} & {}^{i}\exists x\phi & [\text{I}\exists, \text{m}] \end{array}$$

Variable z must count as declared and in i at line m.

(We use $[z/x]\phi$ to indicate the formula obtained from ϕ by replacing all free occurrences of x in ϕ by z. Variable z must be free for x in ϕ.)

∃-Elimination (E∃)

$$\begin{array}{lll} & \vdots & \\ \text{l.} & {}^{i}\exists x\phi & \\ & \vdots & \\ \text{m.} & [z/x]\phi & [\text{ass.}] \\ & \vdots & \\ \text{n-1.} & \psi & \\ \hline \text{n.} & \psi & [\text{E}\exists, \text{l}] \end{array}$$

Variable z may not occur free in any assumptions, ${}^{i}\exists x\phi$ or ψ. It counts as declared and in i from line m to n.

It is assumed that there is default context ^{0} that subsumes the others, so that ${}^{i}\forall x\,{}^{0}\exists y\, x{=}y$, for any i. It is also assumed that, in the extensional format, free variables count as declared in this default context. In the model-theory, quantifiers are interpreted as ranging over contextually given subsets of a default domain of interpretation.

2. An Intensional Format

In the linguistic and philosophical literature it has repeatedly been argued that contextual relativization is of an intensional nature. (Recanati 1996; Reimer 1998, a.m.o.) The contextually relativized derivation rules will therefore be imported in a, mostly standard, modal framework. We here assume a most 'universal' modality, addressed by the default modal operators $\Diamond^0$ and $\Box^0$, which is assumed to be S5 and which is also assumed to subsume the other modalities, so that, in general $\Diamond^k\phi \rightarrow \Diamond^0\phi$. Since we do not consider it a logical principle that everything necessarily exists, free variables may on occasion be understood to fail a referent. However, atomic predications, including identity statements, are taken to entail the existence of a referent of the terms involved, even if their negations do not. We, in other words, adopt a *negative free logic* here. (Nolt 2020) Some convenient notations are also defined here.

E-Introduction (IE)

$$\begin{array}{ll} & \vdots \\ \text{m.} & AT(t) \\ & \vdots \\ \text{n.} & \exists z\ t{=}z \quad [\text{I}\mathsf{E}, \text{m}] \end{array}$$

$$\begin{aligned} {}^{i}\mathsf{E}t &:= {}^{i}\exists z\ t{=}z \\ \Box_x \phi &:= \Box(\mathsf{E}x \to \phi) \\ x \neq y &:= (\mathsf{E}x \wedge \mathsf{E}y \wedge \neg x{=}y) \end{aligned}$$

The rules for identity are standard, but for the fact that substitution of actual identicals is prohibited in modal contexts.

3. Conceptual Windows

In order for a context to serve as a conceptual window, we require it to provide a clear and distinct view on the individuals seen through it.

$$\vdash {}^{i}\forall x\ {}^{0}\mathsf{E}x \tag{E}$$

$$\vdash {}^{i}\forall x {}^{i}\forall y(x{\neq}y \to \Box_x\, x{\neq}y) \tag{D}$$

$$\vdash {}^{i}\forall x \Box({}^{i}\exists x \top \to \mathsf{E}x) \tag{X}$$

What we see through a window exists, the individuals that we see are clearly individuated, and there are no phantom objects.

Contexts that serve as windows not only delimit the extensional domains of quantification, but also serve to determine the ways in which they are presented there. This aligns with the practices in which we actually reason about cross-modal identities. Quine and Kripke, among many others, typically describe a situation in which an individual (Ortcutt, London, ...) is presented, and next observe, or assert, that some agent (Ralph, Pierre, ...) has certain beliefs and wishes concerning that individual, thus conceived. (Quine 1956; Kripke 1979) In all this, it seems, the situation is presented as objective, and in principle accessible to us (actual author, intended reader, and described agent), and so that the agents can be said to have their attitude regarding the individual thus presented. Both aspects can be specified in our framework, neatly, appropriately, and independently.

4. Applications

The description of a situation may take the form of a specification of a context, enumerating the individuals in there, possibly exhaustively, and specifying the properties they are seen to have and the relations they are said to stand in. Such a characterization takes the following general form.

$${}^{i}\exists \vec{x}(\phi(\vec{x}) \wedge \Box_{\vec{x}}\, \psi(\vec{x})) \tag{C}$$

This says that window ^{i} provides a view on the individuals $\vec{x}$ that are ϕ and that are conceived of as being ψ. Context i can be said to provide one of Quine's presentations of Ortcutt, as 'the man seen on the beach', as follows: ${}^{i}\exists x(o{=}x \wedge \Box_x\ {}^{i}\imath z\,(Bz)(x{=}z))$. We can next re-use this context to aptly represent Quine's ascription of the *de re* belief to Ralph (r) that Ortcutt is a spy: ${}^{i}\exists x(o{=}x \wedge \Box^{r}\ Sx)$. Assuming distinctness, the two claims jointly entail that Ralph believes that the man seen on the beach is a spy. The situations of Kripke's Pierre, and his beliefs about London, can be neatly and intuitively presented as well, as well as most of the involved cases known from the literature.

We can also use our windows to neatly represent *knowing-who-is-who*-questions and reports, which have appeared to tantalize linguistics and philosophers. The idea, adapted from (Aloni 2005), is that these questions and reports involve two views, from windows i and j, on what are known to be the same sets of individuals, but so that it is not initially clear which of the individuals seen the one way, is which, when seen the other way. (Think of a list of names of the players, and a view on the field where they play.) This situation, and the fact that agent Rebecca (r) does know *who-is-who*, is most adequately rendered by this characterization: ${}^{i}\forall x^{j}\forall y((\Diamond x{=}y \wedge \Diamond x{\neq}y)\ \textit{and}\ (\Box^{r} x{=}y \vee \Box^{r} x{\neq}y))$.

In the full paper I compare the approach developed here with three currently popular rival approaches, those calling on *conceptual covers*, *concept generators*, and *counterpart relations*. (Aloni 2005; Percus & Sauerland 2003; Charlow & Sharvit 2014; Lewis 1968; Ninan 2018) I show that our framework with conceptual windows can handle whatever the rival approaches can handle, but also that we can do so without having to make any of the dubious metaphysical and/or psychological assumptions that the rivals are committed to.

5. Conclusion

When we reason about modal properties of individuals, and of our beliefs and knowledge of them, we have to take into account the ways in which they are conceived of. (Frege 1892; Quine 1956; Hintikka 1969; Lewis 1968) An independently motivated method of contextually relativized quantification, cast in an intensional setting, gives us precisely this, and nothing else. It may serve to regiment our modal thought and talk without making any kind of metaphysical or cognitive representational assumptions that rival frameworks are committed to.

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Bilateral display calculi

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We develop uniform display calculi for a range of logics that can be construed as being bilateral. Logics we will cover include FDE and related systems LP, K3 (e.g. (Omori and Wansing, 2017)), Nelson's logics N4, N3 (Almukdad and Nelson, 1984), bilattice logic BL (Arieli and Avron, 1996), Wansing's 2-intuitionistic logic 2Int (Wansing, 2016a), connexive logics C, C3, CN, MC (Wansing, 2023) and 2C (Wansing, 2016b) and some others.

We are interested here in a very broad version of bilateralism (perhaps best conveyed in (Smiley, 1996)) which is—unlike Rumfitt's bilateralism (Rumfitt, 2000)—removed from the discussion on proof-theoretic semantics and inferentialism and simply focuses on developing a robust theory of primitive refutation. For this purpose our conception of logic will be that of a multiple-conclusion (for technical reasons) consequence relations over Rumfitt-style signed formulas of the form A^+ for "A is provable" and A^- for "A is refutable". Such bilateral consequence relations will satisfy all of the familiar properties of Scott consequence relations (e.g. (Scott, 1974)) rewritten to accommodate signed formulas. Importantly, we want to develop a theory of primitive refutation in a way that does not depend in any way on the presence of *toggling negation*, which internalizes switching between proofs and refutations.

We use display calculi as a vehicle to develop such a theory. Display calculi are a generalization of sequent calculi due to (Belnap, 1982) obtained be expanding on the number of structural connectives (think comma in the regular sequent calculi) involved in forming sequents. In our case *sequents* are $X \vdash Y$, where X and Y are *structures* built from signed formulas and nullary I using binary structural connectives $\circ$ and $\bullet$. Some advantages of display calculi include a uniform way of proving cut-elimination, a clear separation between structural and non-structural level (which will allow us to make some philosophical remarks on the role of signs) and a close connection to algebras via residuation principles.

The way residuation princples come into play is via a special kind of structural rules called *display rules*, which are at the core of the cut-elimination proof. In our case the display rules (left) intuitively correspond to two residuation principles (right) underlying the bi-intuitionistic logic (Rauszer, 1974):

$$\begin{array}{c} X \vdash Y \bullet Z \\ \hline\hline X \circ Y \vdash Z \\ \hline\hline Y \vdash X \bullet Z \end{array} \qquad \begin{array}{c} X \bullet Y \vdash Z \\ \hline\hline X \vdash Y \circ Z \\ \hline\hline X \bullet Z \vdash Y \end{array} \qquad \mapsto \qquad \begin{array}{c} A \vdash B \to C \\ \hline\hline A \wedge B \vdash C \\ \hline\hline B \vdash A \to C \end{array} \qquad \begin{array}{c} A \leftarrow B \vdash C \\ \hline\hline A \vdash B \vee C \\ \hline\hline A \leftarrow C \vdash B \end{array}$$

As a matter of fact the whole structural part of our base calculus will coincide with that of a display calculus for bi-intuitionistic logic (e.g. (Goré, 2000)). The main difference will be that in our systems each connective c will receive four introduction rules: one for introducing $(A\ c\ B)^\sharp$ into an antecedent position and one for introducing it to the succedent position for each $\sharp \in \{+, -\}$.

The full list of connectives we will consider are contained in the following set:

$$\mathcal{L}_{all} = \{\mathsf{t},\ \mathsf{f},\ \mathsf{b},\ \mathsf{n},\ \sim,\ \wedge,\ \vee,\ \otimes,\ \oplus,\ \to_n,\ \leftarrow_n,\ \prec_n,\ \succ_n,\ \to_c,\ \leftarrow_c,\ \prec_c,\ \succ_c\}$$

where $\{\mathsf{t}, \mathsf{f}, \mathsf{b}, \mathsf{n}, \sim, \wedge, \vee, \otimes, \oplus\}$ are bilattice connectives, $\to_n$ and $\prec_n$ are Nelsonian implication and co-implication, $\to_c$ and $\prec_c$ are connexive implication and co-implication, and the rest are respective backward-looking duals $\leftarrow_n$, $\succ\!\!-_n$, $\leftarrow_c$ and $\succ\!\!-_c$ which are required to interpret $\bullet$ in the antecedent position. With each subset $\mathcal{L} \subseteq \mathcal{L}_{all}$ we will associate a display calculus $\delta\mathcal{L}$ and its bilateral consequence relation $\vdash_{\delta\mathcal{L}}$.

In the full language $\mathcal{L}_{all}$ every connective listed above has two different duals: an *antecedent/succedent* dual corresponding to duality of bi-intuitionistic logic and a *proof/refutation* dual corresponding to inverting the sign in signed formulas. Accordingly we formulate two translations $d_{a/s}$ and $d_{p/r}$ and prove two duality theorems a particular case of which are the following equivalences where $inv(\cdot)$ denotes the operation of inverting the sign:

$$A^\sharp \vdash_{\delta\mathcal{L}_{all}} B^\dagger \quad \text{iff} \quad d_{a/s}(B)^\dagger \vdash_{\delta\mathcal{L}_{all}} d_{a/s}(A)^\sharp \quad \text{iff} \quad d_{p/r}(A)^{inv(\sharp)} \vdash_{\delta\mathcal{L}_{all}} d_{p/r}(B)^{inv(\dagger)}.$$

The first main result of the paper is the completeness between our display calculi and some classes of models. At the most general, the *models* are of the form $\mathcal{M} = \langle W, \leq^+, \leq^-, v\rangle$, where $\leq^+$ and $\leq^-$ are preorders on non-empty W which satisfy some additional conditions and v is a bilateral valuation which maps any signed propsitional variable $p^\sharp$ to a cone with respect to $\leq^\sharp$. Preorders $\leq^+$ and $\leq^-$ are used to model plus-signed and minus-signed implication connectives, respectively. For instance, we have the following satisfaction clauses for connexive implication:

$$\begin{array}{lll} \mathcal{M}, x \vDash (A \to_c B)^+ & \text{iff} & \forall y\,(x \leq^+ y \text{ and } \mathcal{M}, y \vDash A^+ \text{ implies } \mathcal{M}, y \vDash B^+); \\ \mathcal{M}, x \vDash (A \to_c B)^- & \text{iff} & \forall y\,(x \leq^- y \text{ and } \mathcal{M}, y \vDash A^+ \text{ implies } \mathcal{M}, y \vDash B^-). \end{array}$$

The basic completeness is obtain via a modification of the canonical model method, where worlds correspond to kinds of signed theories and $\Omega \leq^\sharp \Sigma$ iff $\forall A\ (A^\sharp \in \Omega$ implies $A^\sharp \in \Sigma)$ for two such theories and $\sharp \in \{+, -\}$. The key statement shows equality between bilateral consequence relations induced by the display calculus and by the semantics for every subset of $\mathcal{L}_{all}$. As a matter of fact, at the base level of our display calculus one can replace two preorders with a partial order corresponding to set-theoretic inclusion on bilateral theories. Notably the semantics is what allows us to claim that the systems listed above can be construed as bilateral: it allows for a natural definition of the corresponding bilateral consequence relation.

Where having two preorders becomes critical is when considering some extensions with axioms and structural rules. To this end we first add the following rule:

$$(\mathsf{collapse})\ \frac{W \vdash X \bullet (Y \circ Z)}{W \vdash (X \bullet Y) \circ Z}$$

Over bi-intuitionistic display calculus thus rule corresponds to adding the generalized law of excluded middle $p \vee (p \to q)$ and, accordingly, to collapsing partial order into equality. Over our bilateral display calculus it has a very different impact depending on which connectives are present in the language. For instance, in the presence of Nelsonian implication this rule corresponds to collapsing $\leq^+$ into an equivalence relation:

$$\forall x, y\ (x \leq^+ y \text{ implies } y \leq^+ x).$$

We investigate the effect this rule has in the presence of different connectives in $\mathcal{L}_{all}$. The addition of this rule allows us to express systems like BL and MC.

To accommodate systems like LP, K3, N3 and C3 we also consider the additions of the following axioms, which can be seen as variants of Rumfitt's *co-ordination princples*:

$$(\mathsf{exp})\quad p^+ \circ p^- \vdash I; \qquad\qquad (\mathsf{em})\quad I \vdash p^+ \circ p^-.$$

These correspond to bilateral versions of the law of non-contradiction and the law of excluded middle, respectively. As it turns out, in general the addition of one of these axioms results in losing the structurality property of the corresponding bilateral consequence relation, since they do not generalize to arbitrary formulas for every subset of $\mathcal{L}_{all}$. We say that a connective c *is not in conflict with* (exp) *or* (em) if the corresponding sequent $(A\ c\ B)^+ \circ (A\ c\ B)^- \vdash I$ or $I \vdash (A\ c\ B)^+ \circ (A\ c\ B)^-$ is derivable in $\delta\mathcal{L} + (\mathsf{exp})$ or $\delta\mathcal{L} + (\mathsf{em})$, respectively. We investigate which connectives are not in conflict with one or both of the axioms and extend our completeness result to appropriate languages. As it turns out the presence of (collapse) changes the picture for some of the connectives. For such languages (exp) corresponds to the condition that $\mathcal{M}, x \nvDash A^+$ or $\mathcal{M}, x \nvDash A^-$ and (em) corresponds to the condition that $\mathcal{M}, x \vDash A^+$ or $\mathcal{M}, x \vDash A^-$ for every model and every formula A.

For some languages we can establish a stronger connection between a display calculus and a bilateral consequence relation induced by the semantics. We say that a display calculus $\delta\mathsf{L}$ is an *equivalent display calculus* of $\vDash_{\mathsf{L}}$ if there are translations $a(\cdot)$ and $s(\cdot)$ from structures to signed formulas such that $X \vdash Y$ is derivable in $\delta\mathsf{L}$ iff $a(X) \vDash_{\mathsf{L}} s(Y)$. This intuitively means that the logic can express everything the display calculus can (whereas the usual completeness corresponds to the inverse implication). In that this notion is clearly quite similar to that of equivalent algebraic semantics (e.g. (Font, 2016)). In the paper we investigate which subsets of $\mathcal{L}_{all}$ induce equivalent display calculi.

Finally, we show how some unilateral systems can be recovered from our calculi.

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Aspects of theoretical synonymy

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In this talk I will discuss the adequacy of prominent relations of theory equivalence as relations of theoretical synonymy. A relation of theoretical synonymy should characterize the equivalence of formal theories with regard to their meaning or content. In logic and the philosophy of science, the relations of definitional equivalence or its generalization, bi-interpretability, are predominantly used for this purpose. However, recent research in mathematical logic raises concerns that certain bi-interpretable theories should not be understood as synonymous.[1] I will argue for a similar skepticism about the adequacy of these relations, but from a more general perspective. To do so, I will introduce five general principles of theoretical synonymy that seem intuitively plausible but are not satisfied by definitional equivalence or bi-interpretability. In particular, I will consider the following aspects of theoretical synonymy:

- Equivalence
- Compositionality
- Decompositionality
- Partial Order
- Sentential Synonymy

Equivalence

Fundamentally, the relation of synonymy between theories should be an equivalence relation. Two theories are said to be definitional equivalent in case they have a common consistent definitional extension. An example is given that shows that such a relation is not transitive for arbitrary languages and transitive but not reflexive for disjoint languages. So despite its name, definitional equivalence is not an equivalence relation.

For this reason, it is better to consider the relation of bi-interpretability. Two theories are bi-interpretable if they are mutually interpretable by translations that are provably inverse to each other.[2] Bi-interpretability is an equivalence relation and, as shown by Friedman and Visser (2014), the relation coincides with definitional equivalence for a large class of theories.

Compositionality

A principle of compositionality can be motivated for synonymous theories: If theories S and T are synonymous, as well as S' and T', then $S \cup S'$ and $T \cup T'$ (if consistent) should also be synonymous with each other. We discuss examples of theories where this principle is violated for bi-interpretability as synonymy.

[1] See e.g. Theorem 24 in Enayat and Lelyk (2024) and their discussion of it in section 5.1.
[2] See (Button and Walsh, 2018, § 5.4) for an exact definition.

Decompositionality

Accordingly, we also discuss a principle of decompositionality: If two theories S and T are synonymous with each other, then for every subtheory S' of S there exists a subtheory T' of T such that S' and T' are synonymous with each other. Again, we discuss some examples of theories S and T that are bi-interpretable but not every subtheory of S corresponds to a bi-interpretable subtheory of T.

Partial Order

Another important aspect of theoretical synonymy concerns the idea that an equivalence relation always has a corresponding non-trivial partial order. Thereby I mean, for a synonymy relation there is a relation $<$ such that: S is synonymous to T if and only if $S < T$ and $T < S$ and $<$ is reflexive, transitive, and $S < T$ or $T < S$ does not already imply the synonymy of S and T. For logical equivalence of theories, the corresponding partial order is simply the subtheory relation. For theoretical synonymy, $<$ would be something as a meaning-preserving reduction of theories or a content-wise reduction.
I will discuss some reasons why it fails to define an obvious partial order to bi-interpretability. Rather, I will strongly suggest that there is no such corresponding partial order. This provides another reason to question the adequacy of bi-interpretability as a relation of theoretical synonymy.

Sentential Synonymy

Finally, I will consider the relationship between theoretical synonymy and sentential synonymy. I suggest that if two theories are synonymous to each other, then there must also hold some relation of synonymy for their theorems. Suppose two theories S and T are bi-interpretable by means of translations f and g. For a relation of theoretical synonymy it should now hold in particular that after the translation of an S-theorem ϕ into a T-theorem $f(\phi)$ and back into an S-theorem $g \circ f(\phi)$, a relation of synonymy (relative to S) exists between ϕ and $g \circ f(\phi)$.

I will consider examples of bi-interpretable theories for which this does not appear to be satisfied for any f and g. My argument for this will rely on further principles of sentential synonymy, in particular that a non-logically true sentence can never be synonymous with a logically true sentence.

All examples considered are of low logical complexity. On the one hand, this helps to quickly falsify the principles mentioned, and on the other hand, it quickly shows how relevant restrictions could look like that lead to the principles being fulfilled.

Finally, I will briefly outline some modifications of bi-interpretability, based, among other things, on a suggestion from van Benthem (1982).

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Vector spaces as topic structures for topic sensitive intensional modals

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A relatively recent trend in logic is the development of logics with *topic-sensitive intensional modalities* (TSIMs). This approach uses a two component (2C) semantics, wherein the "meaning" of a sentence can be divided into two distinct parts: its truth condition and its topic. The 2C semantics has been championed by Franz Berto and co-authors in several papers and books, including Berto (2019, 2022). Existing approaches to 2C semantics typically give a semi-lattice for a topic of structure: for every pair of topics, there is the fusion of those topics. This topic structure is limited by the properties of a semi-lattice. On the other hand, there is a natural approach to modelling topics that is motivated by the semantic (read: topical) relations between the vectors representing words in in Large Language Models (LLMs). In particular, the idea comes from the area of *vector space models* (e.g., see Bengio et al. (2003); Schwenk (2007)). In this paper, we motivate vector spaces as structures for topic by (i) examining the use of vector spaces in LLMs, (ii) replacing semi-lattices by vector spaces in the hyperintensional conditional logic of Özgün and Berto (2021), (iii) argue that the vagueness of "on topic" had a natural representation using vector spaces, and (iv) show that vector spaces (as opposed to mere semi-lattices) excel at representing disagreements between agents grounded in their topic assignments.

Vector space models model words—syntactic tokens in a language—with vectors. These are purely syntactic representations of purely syntactic words. However, the vector space models that result from training LLMs often result in vector spaces that model semantic relations between vectors that, I will claim in particular, model the semantic relations between topic. Reading $+$ as combining vectors and $-$ as subtracting vectors (or taking away the meaning of a vector), we find relations as the following:

> For example, if we denote the vector for word i as x_i, and focus on the singular/plural relation, we observe that $x_{apple} - x_{apples} \equiv x_{car} - x_{cars}$, $x_{family} - x_{families} \equiv x_{car} - x_{cars}$, and so on. (Mikolov et al., 2013a, p. 746)
>
> Somewhat surprisingly, it was found that similarity of word representations goes beyond simple syntactic regularities. Using a word offset technique where simple algebraic operations are performed on the word vectors, it was shown for example that *vector*("King")-*vector*("Man")+*vector*("Woman") results in a vector that is closest to the vector representation of the word *queen*. (Mokolov et al., 2013, p. 2)

These relations between vectors are that kinds of relations we may want to capture between topics. Moreover, these relations are more fine-grained and various than available merely by taking a semi-lattice of topics.

The hyperintensional conditional belief logic of Özgün and Berto (2021) gives a 2 component semantics. We develop an alternative semantics using vector spaces for topics. The language of $\mathfrak{L}_{CHB}$, or well-formed formulas (hereby wff), is defined in Backus–Naur form as follows:

$$\phi ::= p|\top|\neg\phi|\phi \vee \phi|\Box\phi|[\geq]\phi|\mathbb{B}^{\phi}\phi$$

where $\mathbb{B}$ is a conditional belief operator, $\Box$ is an **S5** necessity operator, and $[\geq]$ an operator for 'safe belief'.

Definition 0.1 (Topic Assignments)**.** Suppose that V is a vector space $\mathbb{R}^n$ with standard basis B. Let $\mathfrak{b}$ designate a possibly-zero vector in V. A *topic mapping* (relative to $\mathfrak{b}$) is a map t : wff $\longrightarrow V$ is obtained from a base assignment to atomic propositions $\hat{t}$: atoms $\longrightarrow V$ (such that (1) for no atomic p, $\hat{t}(p) = 0$, and (2) every vector-as-list $\hat{t}(p)$ contains no negative reals) and extended by setting $t(\top) = \mathfrak{b}$ and

> $t(\mathcal{A}) = t(p_1) + \cdots + t(p_n)$, where $p_1, \ldots p_n \in$ Atoms $\cup \{\top\}$ are the atomic formulas with an occurrence in $\mathcal{A}$.

Condition (1) corresponds to the philosophical nicety that every formula has a non-null topic. Condition 2, merely formal, can be motivated by interpreting the positive directions as positive topics, always keeping negative for subtraction (i.e., removing parts of topics). Neither conditions are required. In Özgün and Berto (2021), every formula is assigned a topic as in (1).

Definition 0.2. A *frame for* $\mathfrak{L}_{CHB}$ $\mathfrak{F} = \langle W, \geq, V, \mathfrak{b}, t \rangle$ such that W is a non-empty set, $\geq \subseteq W \times W$ is a well-pre-order, V is an n-dimensional vector space, $\mathfrak{b} \in V$, and t is a topic assignment from wff into V. A *model for* $\mathfrak{L}_{CHB}$ (or *topic-sensitive plausibility model* (tsp-model)) is a frame for $\mathfrak{L}_{CHB}$, $\mathfrak{F}$, and a valuation of atomics $\mathbf{v}$: atom $\longrightarrow \wp(W)$ extended to a full valuation $\vDash$ given by the condition in Özgün and Berto (2021) except: ($|\mathcal{A}| =_{df} \{w \in W \mid w \vDash \mathcal{A}\}$)

1. $w \vDash \mathbb{B}^{\mathcal{A}}\mathcal{B}$ iff (a) $Min_{\geq}(|\mathcal{A}|) \subseteq |\mathcal{B}|$ and (b) $t(\mathcal{B}) \in V[(\mathfrak{b} + t(\mathcal{A}))]$

The condition (b), that $t(\mathcal{B}) \in V[(\mathfrak{b} + t(\mathcal{A}))]$, means that the vector assign to $\mathcal{B}$ lives in the vector space that is the smallest standard subspace of V containing $\mathfrak{b} + t(\mathcal{A})$. Condition (a) is the requirement that all the most plausible $\mathcal{A}$ worlds are $\mathcal{B}$ worlds. In Özgün and Berto (2021), the condition is as follows:

1.′ $w \vDash \mathbb{B}^{\mathcal{A}}\mathcal{B}$ iff (a) $Min_{\geq}(|\mathcal{A}|) \subseteq |\mathcal{B}|$ and (b) $t(\mathcal{B}) \leq \mathfrak{b} \oplus t(\mathcal{A})$

where $\oplus$ is the topic fusion and $|leq$ is the corresponding partial order on the topic semi-lattice. Note that (a) remains the same. We have only changed (b).

We construct a canonical model with a canonical vector space and prove the following.

Theorem 0.3. The logic $\mathfrak{L}_{CHB}$ is sound and complete with respect to the class of tsp-models.

The proof follows Özgün and Berto (2021). First we construct a quasi tsp-model which is not guaranteed to have a finite set W or be well-ordered. A filtration argument shows that the models have a W-finite model property where the relevant filtrations produce models with finite W and finitely generated vector spaces with a finite set of standard subspaces.

Having shown the plausibility of vector spaces through this emulation of the models of $\mathfrak{L}_{CHB}$ in Özgün and Berto (2021), we discuss the unique advantages of using vector spaces in:

1. The modelling of $\mathfrak{L}_{CHB}$ using a real-valued 'close enough' measure of how on topic a formula is.

2. A basic approach for modelling degrees of disagreement on topics

3. A small selection of natural topic-based connectives and their interpretations.

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Characterization of axiomatic extensions of Łukasiewicz unbound logic

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In this talk we will discuss axiomatic extensions of Łukasiewicz unbound logic and give an axiomatization and characterization of them.

Łukasiewicz logic in its infinitely-valued version was introduced by Łukasiewicz and Tarski [14] in 1930 and since then it was proved to be one of the most prominent non-classical logics. This logic is by itself a member of the family of many-valued logics often used to model some aspects of vagueness. Also, it has deep connections with other areas of mathematics such as continuous model theory, error-correcting codes, geometry, algebraic probability theory, etc. [3, 6, 9, 11].

Abelian logic is a well-known contraclassical paraconsistent logic. This logic was independently introduced by Meyer and Slaney [10] and by Casari [2] and it is also called the logic of Abelian ℓ-groups [1] or Abelian Group Logic [12]. This terminology follows from the fact that the matrix models of Abelian logic consist of Abelian ℓ-groups and their positive cones as filters of designated elements (there is also a version of Abelian logic in which the only designated element is the neutral element of the group, which will not be considered here).

Łukasiewicz unbound logic was introduced (but not named) in [4] as a generalization of Łukasiewicz logic. In addition to philosophical and linguistic motivations, this logic can also be motivated purely syntactically, in that the connectives of the unbound Łukasiewicz logic can be seen as an untruncated version of the connectives of the standard Łukasiewicz logic. An axiomatization of this logic can be found in [5].

The varieties of the MV-algebra, classified by Komori in [8], correspond to relative subvarieties of negatively pointed Abelian ℓ-groups, as shown by Young in [13] via the Mundici functor (see [3] for a definition of the Mundici functor). This classification carries over to axiomatic extensions of Łukasiewicz unbound logic.

Although this classification follows from Komori's result using the Mundici functor, we can prove it only using pointed abelian ℓ-groups. This allows us to give a nicer proof of this classification, and in addition, as a corollary, we obtain the classical Komori classification using the Mundici functor.

The class of models of Łukasiewicz unbound logic is a quasivariety defined as $\mathbf{ISPP_U}(\mathbf{R_{-1}})$. In the lattice of relative subvarieties the join irreducible members are the ones generated by Abelian ℓ-groups $\mathbf{Z}_n$ or $\mathbf{Z}_n \overrightarrow{\times} \mathbf{Z}_0$, where $n < 0$. Therefore our goal will be to give an equational description of relative subvarieties generated by $\mathbf{Z}_n$ and $\mathbf{Z}_n \overrightarrow{\times} \mathbf{Z}_0$.

We use the following three equations:

$$n \cdot (x \wedge -x) \leq f \qquad (\mathrm{rank}_n)$$

$$(3n \cdot x - f) \vee (f - n \cdot x) \geq 0. \qquad (\mathrm{rank}^*_n)$$

$$n \cdot ((p \cdot x - f) \wedge (f - p \cdot x)) \leq f \qquad (e_{p,n})$$

Using these three equations we can characterize relative subquasivarieties generated by $\mathbf{Z}_n$ and $\mathbf{Z}_n \overrightarrow{\times} \mathbf{Z}_0$.

Lemma 1. *The relative subquasivariety generated by* $\mathbf{Z}_n$ *is axiomatized by equations* $rank_n$ *and* $e_{p,n}$*, where* p *are primes such that* $p < |n|$.

Lemma 2. *The relative subquasivariety generated by* $\mathbf{Z}_n \overrightarrow{\times} \mathbf{Z}_0$ *is axiomatized by equations* $rank^*_{n+1}$ *and* $e_{p,n}$*, where* p *are primes such that* $p < |n|$.

Moreover, we can also prove that all semilinear finite extensions of Łukasiewicz unbound logic are in bijective correspondence with semilinear finite extensions of Łukasiewicz logic. Semantically, it is known that the semilinear finite extensions of Łukasiewicz unbound logic correspond to Gispert's classification of semilinear finite extensions of Łukasiewicz logic from [7]. However, further study is needed for concrete axiomatization.

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Question-sensitive logic for conditional intention

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1 Introduction

Since the foundational paper of (Cohen, Levesque, 1990) there is an ongoing work on developing a formal logic of intention[1]. In the meanwhile, philosophers of action and legal theorists are focusing more on *conditional* intention[2]. Broadly, conditional intention is a contingent commitment to a course of actions, which depends on a certain condition. For example, if agent is *unconditionally* intends to φ, then she is committed to do so no matter what; but if she intends to φ *in case of* ψ, then the agent's commitment is to φ when she is sure that ψ holds. Conditional intentions are very common. Some even argue that almost all real world intentions are conditional in nature: " *Often even an emphatic unconditional profession of intention such as 'I intend to φ at all costs' or 'I will φ no matter what' is not to be taken literally given that one is not really willing to φ at the price of losing one's life or making Heavens fall. Most intentions appear to be conditional in their 'deep structure' even when the conditions are not explicitly stated"* (Ferrero, 2009, p.700).

To our knowledge, there is still no formal logic of conditional intention. Our paper addresses this gap. First, we outline the conceptual grounds to stress our requirements for any logic of conditional intention. Then, we present a formal language and a semantical framework that adheres to our intuitions.

2 Conditional intention: conceptual grounds

Conditional intention is a contingent commitment to a certain course of actions, which depends on agent's certainty if the given condition holds. A number of criteria we want the logic of conditional intention to satisfy. First, we want it not to be closed under logical entailment. Second, we want conditional intentions not to be formalized via (necessary) material implication. Third, we want conditional intention to hold only for *some* conditions, not for all of them.

As for the closure under logical entailment, the principle was argued to be too strong for the logic of intention even if we idealize agents as perfect logicians. Agent's intention is limited to her *decision problem*. Some consequences of agent's intention may be irrelevant to her decision problem, so that they shall not be intended even if the agent can make the corresponding logical inference.[3] As for why conditional intention is not expressible via (intended) material implication, neither *de re* (agent intends φ in case of $\psi :=$ agent intends that $\psi \rightarrow \varphi$) nor *de dicto* (agent intends φ in case of $\psi := \psi \rightarrow$ agent intends that φ) interpretations are adequate to what conditional intention means[4]. Finally, agent may intend to φ in case of ψ only

[1] See the introduction in (Meyer et. al, 2015) for the brief overview of that work.

[2] See introduction to (Ludwig, 2015) for the overview of conditional intention research.

[3] For the detailed arguments against the closure under logical entailment in the context of intention, see (Beddor, Goldstein, 2023).

[4] See (Ferrero, 2009) for the elaborate argument.

if whether it is ψ or not is important in the context of the decision problem the agent is facing, so that agent should not necessarily have a conditional intention defined for any proposition as a condition[5].

3 Formal framework

Given some countable set of propositional variables $Var = \{p_1, p_2, \ldots\}$, we define the formal language of conditional intention logic $\mathscr{L}^{\Rightarrow}$:

$$\mathscr{L}^{\Rightarrow} \ni \varphi := p \,|\, \neg\varphi \,|\, (\varphi \vee \varphi) \,|\, \Box\varphi \,|\, Int^{\varphi}\varphi$$

where $p \in Var$. The reading of all Boolean connectives and constants is standard, $\Box$ is a universal modality, meaning alethic necessity. $Int^{\psi}\varphi$ stands for *"agent intends that φ in case of ψ"*. We will write simply $Int\varphi$ instead of $Int^{\top}\varphi$.

As for the semantics for $\mathscr{L}^{\Rightarrow}$, for every condition A we are going to formalize the decision problem "*what to do in case of* A?" as a partition of the logical space. Then, an intention is a (partial) solution to the decision problem together with all of its relevant consequnces.[6]

Definition 3.1 (Conditional intention model)**.** A frame $F \in \mathbf{F}^{\Rightarrow}$ is a conditional intention frame iff $F = (W, N, I)$, where:

1. $W \neq \emptyset$ is a non-empty set of *practically* possible worlds. By practical possibility we mean that worlds of W are considered possible by the agent with respect to actions that the agent can perform. If a world is logically possible but would not be real no matter what the agent had done, it is not practically possible. For example, whilst it is logically possible that the agent in question has never been born, it is not practically possible;
2. $N : (W \times 2^W) \to 2^{2^W}$ is a *partial* function, which takes a world w, a proposition A and returns a neighborhood set $N^A(w) \subseteq 2^W$. $N^A(w)$ represents a decision problem "*What to do in case of A?*", where elements of $N^A(w)$ are solutions: if $X \in N^A(w)$, then bringing it about that X is a (partial) solution to the given decision problem. $N^A(w)$ is the set such that $\{X \cap A \,|\, X \in N^A(w)\}$ is a partition of A closed under taking unions.Three constraints:
   (a) if $X \in N^A(w)$, then $X \cap A \neq \emptyset$: potential solutions are consistent;
   (b) If $N^A(w)$ is defined, $N^B(w)$ is defined and $A \cap B \neq \emptyset$, then $N^{A\cap B}(w)$ is defined as well. This condition ensures that it is possible to agglomerate decision problems;
   (c) $N^A(w)$ is defined iff $W \in N^A(w)$: the minimal solution agent has is to bring it about that W;
3. $I : (W \times 2^W) \to 2^{2^W}$ is a partial neighborhood function, such that $I^A(w) = \uparrow X \cap N^A(w)$ for some $X \in N^A(w)$[7]. If X is the solution to $N^A(w)$ that the agent intends to implement,then $\uparrow X \cap N^A(w)$ is the set of propositions that the agent intends to force in case of A. In other words, $I^A(w)$ are the solution the agent is to implement in case of A and all of its relevant consequences. I has the next additional properties:

[5] See (Ludwig, 2015, Sections 2-3)

[6] We follow a tradition of representing decision problems as partitions that originates from (Groenendijk, Stokhof, 1984) and is used for unconditional intentions in (Beddor, Goldstein, 2023). In that light, an expression φ is relevant to the decision problem iff a truth set of the expression $[\![\varphi]\!]$ is equal to some *partial* solution of the problem: a union of some solutions, i.e. the equivalence classes of the corresponding partition.

[7] For any $X \subseteq W$, $\uparrow X = \{Y \subseteq W \,|\, X \subseteq Y\}$.

(a) Given that I^A and $I^{A\cap B}$ is defined, $\bigcap I^A(w)\cap A\cap B\neq\emptyset\rightarrow I^A(w)\subseteq I^{A\cap B}(w)$, i.e. if all that one intends given A is possible to realise in case of $A\wedge B$, one should intend it in case of $A\wedge B$;

(b) $(X\in I^A(w)\wedge Y\in I^{A\cap X}(w))\rightarrow Y\in I^A(w)$, i.e. if one intends X and intends Y given X, then one intends Y as well;

As usual, any frame $F=(W,N,I)$ can be extended to a model $M=(F,V)$ with a valuation function $V:Var\rightarrow 2^W$, where $V(p)$ is the set of worlds, where proposition p is true in the given model.

Definition 3.2 (Semantics clauses for $\mathscr{L}^{\Rightarrow}$)**.** Given any model $M=(W,N,I,V)$ and any world $w\in W$, we recursively define $\models$ relation as folllows:

$$
\begin{aligned}
&M,w\models p\Leftrightarrow w\in V(p)\\
&M,w\models\neg\varphi\Leftrightarrow M,w\not\models\varphi\\
&M,w\models\varphi\vee\psi\Leftrightarrow M,w\models\varphi \text{ or } M,w\models\psi\\
&M,w\models\Box\varphi\Leftrightarrow\forall v\in W:M,v\models\varphi\\
&M,w\models Int^{\varphi}\psi\Leftrightarrow([\![\varphi]\!]_M,[\![\psi]\!]_M)\in I(w)
\end{aligned}
$$

where for any $\varphi\in\mathscr{L}^{\Rightarrow}$, $[\![\varphi]\!]_M=\{w\in W\mid M,w\models\varphi\}$.

A number of interesting (in)validities:

- $Int^{\psi}\varphi\wedge\Box(\varphi\rightarrow\vartheta)\not\models Int^{\psi}\theta$ closure under logical entailment fails;
- $\not\models Int^{\psi}\varphi\leftrightarrow Int(\psi\rightarrow\varphi)$, $\not\models Int^{\psi}\varphi\leftrightarrow(\psi\rightarrow Int\varphi)$: conditional intention cannot be reduced to the unconditional one via (necessary) material implication;
- $Int^{\psi}\varphi\wedge Int^{\vartheta}\varepsilon\not\models Int^{(\psi\wedge\vartheta)}(\varphi\wedge\varepsilon)$: unrestricted agglomeration on both antecedents and consequent fails;
- $Int^{\varphi}\top\wedge Int^{\psi}\top\wedge\Diamond(\varphi\wedge\psi)\models Int^{(\varphi\wedge\psi)}\top$: if agent minds what to do in case of φ and what to do in case of ψ, while φ and ψ are consistent, agent should also mind how to agglomerate her conditional intentions in case of φ and ψ;

During the talk, we are going to address the questions of metalogical properties of the framework (sound and complete axiom system, decidability) and connections with intention revision and information dynamics.

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Pseudo-Scotus on the liar paradox and on Aristotle's theses

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Pseudo-Scotus, the unknown author of the *Questions on the Sophistical Refutations and the Questions on the Prior Analytics* published in Duns Scotus (1891), dealt with the Liar paradox primarily in the variant

[1] 'I say something false',

where it is assumed that the speaker says nothing else but [1]. In order to evaluate the conclusiveness of the arguments for and against [1]'s truth, the Pseudo-Scot discusses the preliminary question whether in proposition

[2] 'Every proposition is false'

the term 'proposition' "supposits" also for the entire proposition in which it occurs. In accordance with the principle "*dici de omni*" (which was apparently defended already by Aristotle), the Pseudo-Scot holds that [2] "says" that itself is false. Therefore, [2] may be regarded as a proposition which violates a basic principle of connexive logic, viz., "Aristotle's Thesis"

[3] $\neg(p \to \neg p)$.

Similarly,

[4] 'Every proposition is true'

leads to a violation of "Boethius's Thesis",

[5] $\neg((p \to q) \wedge (p \to \neg q))$,

because [4] not only maintains itself to be true, but it also entails that the negation of [4] is true. Thus, the Pseudo-Scot recognized that propositions such as [2] and [4] *refute themselves*, i.e., the assumption that they are true entails that they are false, in contradiction to [3].

Now, the Liar paradox [1] appears to be not only self-refuting, but also "self-verifying" in the sense that the assumption that [1] is *false* entails that [1] is *true*. The Pseudo-Scot concedes the correctness of the pro-argument according to which, if [1] is true, and if the speaker says nothing else but [1], [1] must be false. But he denies—or at least confines—the correctness of the contra-argument by claiming that [1] is not *simply* true ('verus simpliciter'), but true only *in a certain respect* ('verus secundum quid'). Somewhat more exactly (but not precisely enough), the truth "secundum quid" shall consist in the fact that a "true" act of saying is exercised with respect of a false "oration". As Spade and Read put it in (2021), "The author of the *Questions on the Sophistical Refutations* [i.e., Pseudo-Scotus ...], thought that what the liar is really doing (the "exercised act") is speaking the truth. In order to avoid the paradox, this theory would seem to be committed to saying that the exercised act and the signified act are two distinct acts".

In connection with the search for a correct definition of a sound inference ("bona consequentia"), the Pseudo-Scot discussed the interesting sophism

[6] 'God exists, therefore this inference is invalid'.

As is evident from the works of *Albert of Saxony*, this sophism may be regarded as a variant of the Liar paradox. In Albert von Sachen (2010), pp. 1110-1160, the "ordinary" version [1] is successively transformed into "hypothetical" versions such as 'God exists and some conjunction is false', 'A man is a donkey or some disjunction is false', and 'If God exists, then some implication is false', which may finally be converted into [6].

As was pointed out in Mates (1985) and in Read (1979), however, *Pseudo-Scotus* failed to recognize that [6] is a variant of the deeply paradoxical Liar [1]; he considered [6] as a mere sophism which only threatens the correctness of a familiar definition of a "consequentia bona". This definition says

[7] '$(p \Rightarrow q)$ is valid iff it is impossible that p be true and yet q be false' (where it must perhaps be further required that p and q are "formed together").

The Pseudo-Scot tried to solve the problems generated by [6] by maintaining that definition [7] is correct with the only exception where q itself "signifies" that $(p \Rightarrow q)$ is invalid. This rather ad hoc "solution" of the Liar paradox is not very convincing, however.

As regards the basic laws of connexive logic, the Pseudo-Scot saw very clearly that the very *definition* [7] gives rise to the validity of the principles "Ex impossibili quodlibet" and "Necessarium ad quodlibet":

[8] If $\neg\Diamond p$, then $(p \Rightarrow q)$, for any q,

[9] If $\Box q$, then $(p \Rightarrow q)$, for any p.

For, if p is impossible, or if q is necessary, i.e., if $\neg q$ is impossible, then *a fortiori* the conjunction $(p \wedge \neg q)$ is impossible, so that, on account of [7], $(p \Rightarrow q)$ becomes true. The fact that a necessary consequent q follows from *any* antecedent p entails, however, that q follows both from p and from $\neg p$, *in contradiction to Aristotle's Thesis*. Furthermore, the Pseudo-Scot was well aware of the fact that the more specific principle "Ex *contradictione* quodlibet", e.g., in the form

[10] $(p \wedge \neg p) \Rightarrow q$,

can be *formally proved* by means of a few logical principles including so-called "disjunctive syllogism". Therefore, the self-contradictory conjunction $(p \wedge \neg p)$ logically entails *any* consequent q, and in particular it entails *both* conjuncts p and $\neg p$, *in contradiction to Boethius's Thesis* [5]. Furthermore, the Pseudo-Scot pointed out that, independently of principle [9], the tautological disjunction $(p \vee \neg p)$ logically follows from both disjuncts. Thus, "Quaestio III" of the *Questions on Book II of the Prior Analytics* concludes as follows:

> So, it is evident that the rule 'One and the same proposition does not follow from something's being so, and from its not being so' must only be understood [to hold] for simple categorical propositions and only in the sense of formal consequences; for as a material consequence [...] the same follows from both contradictories. Similarly, [...] if the consequent is a disjunction composed of contradictories, the rule is not true. (Duns Scotus (1891), pp. 185-6)

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The syntax and semantics of intuitionistic modal logic with diamond

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Intuitionistic modal logics (IMLs) are the results of the extension of intuitionistic propositional logic ($\mathbf{IPL}$) by means of modalities $\Box$ and $\Diamond$. There are various ways to define IMLs. In many approaches, $\Box$ and $\Diamond$ are normal. In most IMLs, unlike the classical situation in modal logics, the operators $\Diamond$ and $\Box$ are not supposed to be interdefinable. However some relations between $\Diamond$ and $\Box$ are assumed which give rise to different families of IMLs. The most traditional ones are IK introduced by [Fischer Servi [15]] and CK described in [Bellin et al. [1]]. The constructive IMLs CK is obtained from IK by assuming that $\Diamond$ is non-normal and thus dropping some relations between $\Diamond$ and $\Box$ in IK. Both classes of logics received a lot of attentions and were developed by many researchers from the semantic and syntax point of view [16, 2]. The study of the decidablitiy of the validity problems of IMLs can trace back to [Grefe[8]], in which IK is shown to have finite model property and thus is decidable. And the decision problem of IS4 was considered to be an open question in the long term and solved recently in [Girlando et.al[7]]. The $\Diamond$ fragment of IMLs are considered by [Celani and Montangie[5]]. In the same paper topological representation for the Heyting algebras corresponding to these logics are proved. We continue this line of research and consider IMLs with $\Diamond$ denoted by Int$\Diamond$s. This class of logic can be formalized as follows.

Let $\mathbf{Var}$ be a countably infinite set (with typical members called *atoms* and denoted p, q, etc). Let $\mathbf{F}$ be the set (with typical members called *formulas* and denoted φ, ψ, etc) defined by

$$\varphi ::= p \in \mathbf{Var} | \top | \bot | (\varphi \vee \varphi) | (\varphi \wedge \varphi) | (\varphi \to \varphi) | \Diamond \varphi$$

We follow the standard rules for omission of the parentheses. For all formulas φ, ψ, we write $\neg\varphi$ and $\varphi \leftrightarrow \psi$ instead of $\varphi \to \bot$ and $(\varphi \to \psi) \wedge (\psi \to \varphi)$.

A Int$\Diamond$ is a set of formulas containing the standard axioms of $\mathbf{IPL}$, closed with respect to the standard inference rules of $\mathbf{IPL}$, containing the axioms

$$(\mathbf{Axiom}\Diamond 1)\ \neg\Diamond\bot, \ \ (\mathbf{Axiom}\Diamond 2)\ \Diamond(p \vee q) \to \Diamond p \vee \Diamond q.$$

and closed with respect to the inference rule

$$(\mathbf{Rule}\Diamond)\ \frac{p \to q}{\Diamond p \to \Diamond q}.$$

We study Int$\Diamond$s by semantic and syntactic approach. In traditional semantic approach, the semantic structures for interpreting intuitionistic modal formulas are typically with two relations, one for the modality and one for the partial order of intuitionistic propositional logic, while we have three relations: two relations for the intuitionistic propositional logic part and one relation for the modality.

Possibility Semantics and Fairtlough-Mendler semantics. Possibility semantics [12] is a generalization of possible world semantics for modal logic, based on partial possibilities instead of complete possible worlds like the ones in standard possible world semantics. In recent years there have been a lot of studies in possibility semantics [9, 11]. Fairtlough-Mendler semantics [6] (FM-semantics for short) can be seen as the possibility semantic counterpart for intuitionistic logic [3, 4], whose dual algebraic structures are complete Heyting algebras which are not necessarily perfect.

In FM-semantics for intuitionistic logic, the FM-frames are of the form $(W, \leq_1, \leq_2)$ such that W is the set of possibilities, $\leq_1$ and $\leq_2$ are partial orders on W such that $\leq_2 \subseteq \leq_1$. Intuitively (see [13, Section 5.4] and [4, Remark 4.29] for more details):

- W is a non-empty set of points which are partial descriptions of information states.
- $x \leq_1 y$ iff every state partially described by y is more informative about the world than some state partially described by x.
- $x \leq_2 y$ iff every state partially described by y is also partially described by x.

At the model level, every propositional variable is interpreted as a refined regular open subset of W which is a special $\leq_1$-upset such that the following two conditions hold:

- Persistence: if w settles p as true, then so does any further refinement $v \geq_1 w$.
- Refinability: if w does not settle p as true, then w can be refined to some $v \geq_1 w$ that settles p as false, so no refinement $u \geq_2 v$ settles p as true.

For intuitionistic logic with a diamond, we add a binary accessibility relation to the FM-frames, and to guarantee that $V(\Diamond\varphi)$ is a refined regular open subset, additional constraints on the interaction between $\leq_1$, $\leq_2$ and R need to be imposed.

In the syntactic approach, we present Int$\Diamond$ and its extensions by Genzten-style calculus by adapting terminology from [14]. We use a structural operation $\circ$ for the connective $\Diamond$. The *set* **FS** *of all formula structures* (with typical members denoted Γ, Δ, etc) is defined by

$$\Gamma ::= A \mid (\Gamma, \Gamma) \mid \circ\Gamma$$

where A ranges over **F**. With the help of $\circ$, $\Diamond$ rules can be presented as in [14] and properties on $\Diamond$ can be described by some structural rules of $\circ$.

Our Results. In our work, we adapt FM-semantics to intuitionistic logic with a diamond modality (Int$\Diamond$) by adding a binary relation for the modality $\Diamond$. We use modal FM-frames of the form $(W, \leq_1, \leq_2, R)$, where $(W, \leq_1, \leq_2)$ is an FM-frame and R is a binary relation on W such that certain interaction conditions hold, and at the model level, propositional variables are also interpreted as refined regular open subsets.

We also make use of the representation results in [10, 13] for complete Heyting algebras to dually represent complete Heyting algebras with completely additive operators as modal FM-frames. Based on the representation results, we also adapt the correspondence results in [17] to Int$\Diamond$.

We present a Genzten-style sequent calculus for Int$\Diamond$ enriched with $ESp_\Diamond$ axioms. The $ESp_\Diamond$ formula is defined as follows.

- A formula is called $Sp^-_\Diamond$ formula if it is generated from **Var** by connectives $\Diamond, \wedge$. A formula is called $Sp^+_\Diamond$ formula if it is generated from **Var** by connectives $\Diamond, \wedge, \vee$.
- A $ESp_\Diamond$ formula is defined recursively as follows
  - A $Sp^-_\Diamond$ formula is a $ESp_\Diamond$ formula,
  - $\varphi \to \psi$ is a $ESp_\Diamond$ formula if φ is a $Sp^+_\Diamond$ formula and ψ is a $ESp_\Diamond$ formula.

Equivalently a $ESp_{\Diamond}$ formula can be written in the form of $\varphi \to \psi$ where φ is a $Sp_{\Diamond}^{+}$ formula and ψ is a $Sp_{\Diamond}^{-}$ formula. Cut elimination for all Int$\Diamond \oplus ESp_{\Diamond}$ are shown. Based on it, the decidability results are obtained for Int$\Diamond \oplus \{\varphi \to \psi | \varphi \to \psi \in SESp_{\Diamond}\}$ where $SESp_{\Diamond} = \{\varphi \to \psi | \varphi = \Diamond^k p \& p \in \mathbf{Var} \& 0 \leq k\} \cup \{\varphi \to \psi | R_{\Diamond}(\varphi) = 1\}$. By $R_{\Diamond}(\varphi) = 1$, we mean that there is no nested $\Diamond$ appearing in φ. Finally we show that the complexity upper bounds of the validity problem of all Int$\Diamond \oplus SESp_{\Diamond}$ are NEXPTIME.

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Recent advances in fundamental logic

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Holliday (2023) introduced a non-classical logic called *fundamental logic*, which captures exactly those properties of the connectives $\wedge, \vee$ and $\neg$ that hold in virtue of their introduction and elimination rules in Fitch's natural deduction system for propositional logic. Fundamental logic is a sublogic of both (the $\rightarrow$-free fragment of) intuitionistic logic and orthologic. The former can be obtained from fundamental logic by adding the *Reiteration* rule to Holliday's Fitch system for fundamental logic, while the second can be obtained by adding the *Double Negation Elimination* rule.

From the algebraic perspective, fundamental logic is the logical counterpart to the variety of fundamental lattices:

Definition 1. *A fundamental lattice is a tuple $(L, \leq, \wedge, \vee, \neg, 0, 1)$ such that $(L, \leq, \wedge, \vee, 0, 1)$ is a bounded lattice and $\neg : L \rightarrow L$ is an antitone map satisfying the following properties:*

- $\neg 1 = 0$*;*
- $a \wedge \neg a = 0$*;*
- $a \leq \neg\neg a$*.*

Since fundamental logic is weaker than both intuitionistic logic and orthologic, fundamental lattices generalize both pseudocomplemented distributive lattices and ortholattices.

There are several reasons to consider a logic weaker than both intuitionistic logic and orthologic. In the mathematical context, fundamental logic can be viewed as a "common ground" for both constructive reasoning and non-distributive reasoning (for example, in the context of quantum mechanics). In the formal semantics of natural language, several examples have been proposed to challenge the unrestricted validity of Double Negation Elimination (e.g., vague predicates) or of the Reiteration rule (e.g., epistemic modals). A defining feature of fundamental logic, however, is that it does not exactly coincide with the "common core" of intuitionistic logic and orthologic, but is rather a weaker system than the intersection of the two.

In this talk based on two joint projects with Wes Holliday and Juan P. Aguilera respectively, I will present some recent results which shed some new light on the relationship between fundamental logic, intuitionistic logic and orthologic.

First, I will discuss two translations of fundamental logic into modal orthologic and modal intuitionistic logic. The first translation is based on the celebrated Gödel translation (Gödel, 1933) of intuitionistic logic into S4, the modal logic of reflexive and transitive Kripke frames, also studied by McKinsey and Tarski (1948). The restriction of this translation to the $\rightarrow$-free fragment of IPC is a map τ inductively defined as follows:

$$\tau(p) = \Box p;$$

$\tau(\neg\phi) = \Box\neg\tau(\phi)$;

$\tau(\phi \wedge \psi) = \tau(\phi) \wedge \tau(\psi)$;

$\tau(\phi \vee \psi) = \tau(\phi) \vee \tau(\psi)$.

As it turns out, this translation also yields an embedding of fundamental logic into $\mathsf{OS4}$, the natural counterpart of $\mathsf{S4}$ in orthomodal logic.

Theorem 1 (Holliday and Massas, 2025)**.** *The Gödel-McKinsey-Tarski translation τ is a full and faithful translation of fundamental logic into* $\mathsf{OS4}$.

A similar result can be obtained by "swapping" the roles of intuitionistic logic and orthologic. Goldblatt (1974) defined the following translation σ from the language of orthologic into the language of modal logic:

$\sigma(p) = \Box\Diamond p$;

$\sigma(\neg\phi) = \Box\neg\sigma(\phi)$;

$\sigma(\phi \wedge \psi) = \sigma(\phi) \wedge \sigma(\psi)$;

$\sigma(\phi \vee \psi) = \Box\Diamond(\sigma(\phi) \vee \sigma(\psi))$.

Goldblatt shows that σ is a full and faithful translation of orthologic into the modal logic KTB of reflexive and symmetric Kripke frames. In order to generalize this result, we define the logic FSTB, a natural counterpart of KTB in the setting of Fischer Servi intuitionistic modal logics (Fischer Servi, 1977).

Definition 2. *The intuitionistic modal logic* FSTB *extends the Fischer-Servi logic* FS *with the following axioms:*

$\Box\phi \vdash \phi$, $\phi \vdash \Diamond\phi$;

$\Diamond\Box\phi \vdash \phi$, $\phi \vdash \Diamond\Box\phi$.

Theorem 2 (Holliday and Massas, 2025)**.** *The Goldblatt translation σ is a full and faithful translation of fundamental logic into* FSTB.

These results establish that fundamental logic is, arguably, both "intuitionistic logic from the viewpoint of orthologic", and "orthologic from the viewpoint of intuitionistic logic".

Lastly, I will discuss the relationship between fundamental logic and orthointuitionistic logic, i.e., the strongest logic contained in both the $\to$-free fragment of intuitionistic logic and orthologic. Although fundamental logic is strictly weaker than orthointuitionistic logic, the latter turns out to have a reasonably simple axiomatization.

Definition 3. *Let* OI *be the smallest consequence relation extending fundamental logic and closed under the following axioms:*

$$\neg\neg p \wedge \neg\neg q \vdash \neg\neg(p \wedge q); \tag{Nu}$$

$$\neg\neg p \wedge q \wedge (r \vee s) \vdash p \vee (q \wedge r) \vee (q \wedge s); \tag{Vi}$$

$$\neg(p \wedge ((q \wedge r) \vee (q \wedge s))) \wedge p \leq (q \wedge (r \vee s)) \vee \neg(q \wedge (r \vee s)). \tag{Cl}$$

Theorem 3 (Aguilera and Massas, 2025)**.** *The logic* OI *is the strongest extension of fundamental logic that is weaker than both orthologic and intuitionistic logic.*

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From scientific philosophy to logic as a science

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In the wake of Quine's famous rejection of the analytic-synthetic distinction, several attempts to alternativelyy justify logical knowledge have been made. More recently, a popular proposal is being discussed under the name of "anti-exceptionalism about logic" (Martin and Hjortland (2024); Williamson (2007); Russell (2022)), a cluster of thesis united by the ideas that logic is revisable, non-foundational and continuous with science. There are different characterizations and variants of the thesis, but all of them are committed in rejecting the traditional epistemology of logic, according to which logical knowledge is a priori, its truths are analytic, general, and necessary, its justification is non-inferential, and its methods are unique. Naturally, a new range of questions open up: if not by their analyticity, how are the basic logical laws justified or adopted? If logical theories can be revised, what relevant data and evidence are needed? What phenomena can logics explain, and what kind of explication is that? If logic is not wholly general, what is its subject matter?

Although these issues are not completely new, they have resurfaced with different motivations and analysis tools. There are two main anti-exceptionalist approaches: continuity with science and tradition rejection. The first captures a loose naturalistic attitude and was characterized by so-called "Quinean claims": (i) gradualism, that there is a continuity in degree between logical and scientific theories; (ii) revisionism, that logic should be revisable on the same grounds as science; and (iii) non-apriorism, that logical truths are not justified solely by a priori evidence (Hjortland (2019)).

The second is a negative thesis that denies at least one of the traditional properties attributed to logic, including, but not limited to: generality, formality, apriority, analyticity, necessity, and non-inferentialism. Depending on how this rejection is carried out, one can obtain either a metaphysical variant in which the subject matter of logic has no privileged or special nature, or an epistemological one, which primarily denies logic a foundational epistemology and is split into yet another two subvariants: evidential and methodological. The former regards the admissible sources of evidence and the latter the methods of theory-selection in logic.

While anti-exceptionalists are openly inspired by Quine, taken as the vanguard of the movement, his ideas are not adopted wholesale. To better grasp the significance of anti-exceptionalism about logic as a viable alternative for the epistemology of logic, I propose a historical venture with a systematic outlook. That is, a historical reconstruction of anti-exceptional ideas about logic - and their developments and transformations - by comparing earlier iterations with contemporary ones, as means to identify the evolving narrative of logic as not special.

The first stop is Quine's naturalism, in which there is an influential rupture with the traditional justification of logical knowledge via the rejection of the analytic-synthetic distinction. Once this is abandoned, logic loses one of its most special features (analyticity) and has to be evaluated on the same grounds as any other branch of scientific knowledge, despite its very privileged position in the web of beliefs. Not only that, the scope of admissible evidence is limited to the natural sciences, an aspect that is still controversial and plays a role in distinguishing modern versions of anti-exceptionalism (Martin and Hjortland (2024)).

An interesting feat of Quine's philosophy is how it was shaped by the influence of both pragmatism and logical empiricism. The connection of these two movements isn't always appreciated, but their similarities are (and were) fairly acknowledged by members of both sides, in fact being decisive for the reception of logical empiricism after its dissolution and internationalization in the 1930s (Misak (2013); Stadler (2015)). For this talk I will focus on the logical empiricist side. While Kant's synthetic *a priori* was rejected as one of the main thesis of the movement, they still maintained the analytic-synthetic distinction, which set them apart from Quine. Nonetheless, they were all united by a scientific attitude towards philosophy, a rejection of epistemic foundationalism and of philosophy-first approaches, as well as in favor of a fallible and revisable epistemology, pretty much anti-exceptional features from today's perspective.

If we push this thread even further, I argue that Frege's anti-psychologism - an apparently obvious choice of exceptionalist - is the outset for modern AEL, emerging within the broader project of scientific philosophy. Put simply, psychologism is a tendency or doctrine in which a given discipline, such as logic, can be epistemically or metaphysically reduced to psychology and to its respective psychological entities and phenomena. Thus the content of logical terms can be understood as descriptions of, or features belonging to, the human mind, making its subject matter dependent on empirical investigation. Maybe surprisingly, psychologism qualifies as an early anti-exceptionalist perspective on logic (Martin and Hjortland (2022)), but this is clearly not what is currently meant by the movement. I argue that the key to understanding this missing link lies in Frege's anti-psychologism, which shaped modern logic while striving to preserve traditional features.

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Cardinality in a paraconsistent and paracomplete set theory

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In Khomskii & Oddsson (2024), a natural formalization of set theory called BZFC was developed in the logic BS4 from (Omori & Waragai, 2011). In the semantics of BS4, truth and falsity are separated, so a statement φ can be true and not false (**t**), false and not true (**f**), both true and false (**b**), or neither true nor false (**n**). The theory BZFC is based on a careful generalization of ZFC, together with an axiom that postulates the existence of non-classical sets.

In BZFC, a set A can be described by its *positive extension* (the collection of all x such that $x \in A$ is true) and *negative extension* (the collection of all x such that $x \in A$ is false), and this can be expressed within the system. However, it turns out to be more appropriate to talk about the complement of the negative extension, i.e., the collection of all x for which the statement "$x \notin A$" is not true. This is because the negative extension is a proper class while its complement is a set.

The positive extension is denoted by $A^!$, and the complement of the negative extension is denoted by $A^?$. Together $A^!$ and $A^?$ completely describe A (Figure 1). Note that A is completely described by how its membership relation acts on the set $rlm(A) := A^! \cup A^?$, called the *realm* of A.

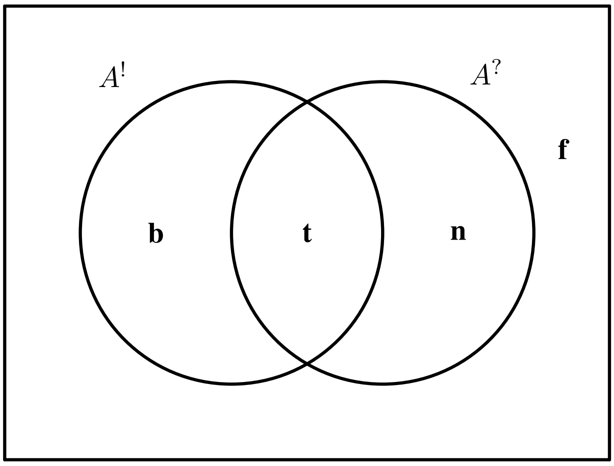

Figure 1: The four truth values of $x \in A$ depending on the boolean combination of $A^!$ and $A^?$.

A set A is called *inconsistent* if there is an element x for which "$x \in A$" gets the truth value **b**, *incomplete* if there is an element x for which "$x \in A$" gets the truth value **n**, and *classical* if it is neither inconsistent nor incomplete, i.e., for all x, "$x \in A$" gets the truth value **t** or **f**.

The central question we aim to answer is this: *How do we measure the size of sets that are inconsistent or incomplete?* Take for example a set A with a unique element a such that "$a \in A$" is true, while for *all* x, including a, "$x \in A$" is false. In other words, A is the set given by $A^! = \{a\}$ and $A^? = \emptyset$. Then A is *inconsistent*, as "$a \in A$" is both true and false. How many elements does A have? On the one hand, we could try to say that A has *zero* elements since everything is a non-member of A. However, this fails to capture that A has an element, namely a. On the other hand, we could try to say that the number of elements in A is *one* since a is the unique element of A. This, however, fails to capture the fact that A has no elements in the sense that everything is a non-member of A. So, both one and zero fail to capture how many elements are in A, and we are forced to admit that we will need a new kind of number to describe the size of A.

As a starting point, we take the following passage from Cantor:

> We call by the name "power" or "cardinal number" of M the general concept which, by means of our active faculty, arises from the aggregate M when we make abstraction of the nature of its various elements m and of the order in which they are given. (Cantor, 1895/1952, p. 85)

While this is not exactly precise, we can use this as a guide in our investigations. First, however, we need to note that while Cantor only talks about the elements of a set, we need to consider what happens on the whole of the realm of a set. So, we will informally think of the cardinality of a set as what remains when we abstract away all the particulars about the elements of the realm of said set. With a bit of finesse, this leads us to the following definition:

Definition 0.1. *We say that two sets A and B have the same cardinality, and write $A \cong B$, if there exists an injection f from $rlm(A)$ such that $f[A] = B$.*

There is also a class of objects called the *cardinal numbers* such that for each set A, there is a unique cardinal $|A|$ such that the following holds:

- $|A| = |B|$ if and only if $A \cong B$
- $|A| \neq |B|$ if and only if $A \ncong B$.

We can now start developing the theory of cardinal arithmetic by letting

$$|A| + |B| := |A \uplus B| \text{ and } |A| \cdot |B| := |A \times B|$$

where $A \uplus B$ denotes the disjoint union of A and B. We also define the constants

$$1 := |\{\emptyset\}|,\ 0 := |\emptyset|,\ \mathfrak{b} := |X| \text{ and } \mathfrak{n} := |Y|,$$

where X and Y are the sets such that "$\emptyset \in X$" and "$\emptyset \in Y$" get the truth values **b** and **n**, respectively.

It turns out that for all A,

$$|A| = |A_{\mathbf{t}}| + |A_{\mathbf{b}}| \cdot \mathfrak{b} + |A_{\mathbf{n}}| \cdot \mathfrak{n},$$

where $A_{\mathbf{t}}$, $A_{\mathbf{b}}$, and $A_{\mathbf{n}}$ are the classical sets of elements such that the statement "$x \in A$" gets the truth value **t**, **b**, and **n**, respectively. So, every cardinal x can be written uniquely as

$$x = x_{\mathbf{t}} + x_{\mathbf{b}} \cdot \mathfrak{b} + x_{\mathbf{n}} \cdot \mathfrak{n}$$

for classical cardinals $x_{\mathbf{t}}$, $x_{\mathbf{b}}$, and $x_{\mathbf{n}}$.

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Hyperintensions as computations

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In this talk, we show that the hyperintensional typed lambda calculus (HTLC) of Fait and Primiero (2021) inspired by transparent intensional logic is equivalent to the computational lambda calculus (CLC) of Moggi (1991) extended by a simple axiom. We demonstrate this by first establishing a link between HTLC and propositional lax logic (PLL) which corresponds to CLC via the Curry-Howard isomorphism. To our knowledge, this connection has not previously been known. Our result puts on solid formal ground a long-held assumption that there is a close connection between the notions of structured hyperintension and computation.

One of the key insights of Moggi's CLC is that reasoning about computations in a uniform way requires a careful distinction between programs (that can have various side effects, e.g., failure to deliver a value) and values. For that purpose, he introduced a new unary operator T for constructing a computational type $T\,A$ that is a type of program that computes values of type A. This is a distinction that is nonexistent in simply typed lambda calculus where the type of a program is the same as the type of its value. Later, it was observed by Benton et al. (1998) that Moggi's CLC corresponds via the Curry-Howard isomorphism to propositional lax logic (PLL) developed by Fairtlough and Mender (1997) which is a constructive modal logic introducing a new modality $\circ$ that combines aspects of both necessity and possibility operator.

There is also another system that is built with the same key insight that computations and values should not be conflated and that is Tichý's transparent intensional logic (TIL, Tichý (1988), Duží et al. (2010)) with hyperintensional procedural semantics. However, unlike Moggi's CLC, which has been extensively studied from many standard semantic perspectives, such as model-theoretic (Fairtlough and Mender (1997)), algebraic (Goldblatt (1981)), or categorical (Moggi (1991)), TIL still remains largely unexplored in this respect. This is due in no small part to its complexity and high expressive power initially intended for analyzing the semantics of natural language. As Berto and Nolan (2021) commented, "the [TIL] approach is less popular than it should be in contemporary semantics, possibly due its resorting to a technical apparatus of typed lambda calculus."

Recently, however, Fait and Primiero (2021) developed a hyperintensional typed lambda calculus (HTLC) which is inspired by TIL but simplifies many of its features and adopts an alternative, proof-theoretic approach. Yet, at the same time, it still retains some of TIL's fundamental principles. Most importantly, it borrows from TIL the idea of Trivialization construction, which is effectively a trivial computation whose input is the same as output, and HTLC treats it as a nullary operator $*$ for constructing hyperintensional types.

At first glance, HTLC might appear to be just a simpler version of TIL, but as we will show, the system is actually quite intriguing. Most notably, a closer look reveals that from a logical point of view, HTLC can be regarded as a propositional lax logic (PLL, Fairtlough and Mender (1997)) extended with an additional axiom (E): $\circ A \to A$, i.e.,

$$\mathsf{HTLC} \;=\; \mathsf{PLL} + (\mathrm{E})$$

as they have the same set of theorems. Furthermore, thanks to the Curry-Howard isomorphism between PLL and CLC mentioned above, this also means that HTLC can be seen as a variant of CLC extended by the same axiom (E), i.e.,

$$\mathsf{HTLC} \;=\; \mathsf{PLL} + (\mathrm{E}) \;=\; \mathsf{CLC} + (\mathrm{E})$$

and thus we also obtain a natural correspondence between hyperintensional types of HTLC and computational types of CLC.

This not only provides an interesting new interpretation of HTLC as a modal logic, specifically as a variant of computational logic in the sense of Benton et al. (1998) but, considering that the key hyperintensional modality $*$ of HTLC was directly inspired by the Trivialization of TIL, it also sheds new light on the Trivialization construction and its dual Double Execution construction and on TIL itself. More specifically, our results suggest that TIL might be actually much closer to the computational lambda calculus CLC of Moggi than it seemed before. On the other hand, this should not be as surprising, since, as mentioned above, both Moggi and Tichý were particularly interested in developing systems that kept computations and values strictly apart.

Finally, as mentioned above, there is also an interesting philosophical implication of this connection. The established link between CLC and HTLC (and TIL), that is, between computations and structured hyperintensions, formally validates a conjecture initially made by Tichý (1986) (p. 526): "The notion of [effective] construction is correlative with a particular algorithmic *computation*."

Traditionally, the notion of hyperintensionality is studied from a logical point of view. For example, one typically introduces a hyperintensional propositional operator H that can render any two logically equivalent propositions, e.g., $A \wedge B$ and $B \wedge A$, distinct, i.e., $H(A \wedge B) \neq H(B \wedge A)$. With HTLC, we will be rather focused on hyperintensionality from a functional point of view. To give an example, assume that we have two (β-)equivalent lambda terms $\mathsf{ap}(\lambda x.b(x), a)$ and $b(a)$ of type A. Analogously to the above, we can then introduce a hyperintensional term operator h that will render them nonequivalent, i.e., $h(\mathsf{ap}(\lambda x.b(x), a)) \neq h(b(a))$, while still retaining the same hyperintensional type $H'(A)$ for both of them. In HTLC, the role of h and H' will be played by triv and $*$, respectively (see below).

Formally, HTLC is a typed lambda calculus extended with a unary term constructor triv and the associated type constructor $*$ (pronounced "star"). We present our variant of HTLC system with a generalized elimination rule for the operator $*$. The most significant departure from Fait and Primiero (2021) is that we will interpret $*$ not as a trivialization operator but as a general execution operator. This is motivated by the fact that the trivialization operator of HTLC is based on the Trivialization construction of TIL which can also be understood as a special case of degenerate Zero Execution construction (Pezlar (2022)).

We will work with a fragment containing only implication ($\rightarrow$) and the general execution operator ($*$) as it will be sufficient to demonstrate all the main points. However, HTLC can easily be extended to include conjunction, disjunction, and negation as well.

The *language* $\mathcal{L}$ of HTLC is built from terms and types. The *terms* of HTLC are generated by the following grammar: $t, s ::= x \mid \lambda x.t \mid \mathsf{ap}(t, s) \mid \mathsf{triv}(t) \mid \mathsf{exe}(t, x.s)$. The *types* of HTLC are generated as follows: $T, S ::= \alpha \mid T \rightarrow S \mid *T$ where α ranges over a countable infinite set of atomic extensional types. The terms and types combine together into typing *judgments* of the general form: $\Gamma \vdash t : T$ where Γ is a context defined as follows: $\Gamma ::= \cdot \mid \Gamma, x : T$. See Fig. 1 for logical, computation, and expansion rules of HTLC. Note on ($*$GE): the notation $x.s$ in $\mathsf{exe}(r, x.s)$ means that the variable x becomes bound in s by exe. In addition to these rules, we also assume the standard structural rules of weakening, exchange, and contraction and the substitution rule. The resulting calculus HTLC has some desirable properties such as (weak) normalization.

For future work, the natural next step would be to further explore the links between (1) HTLC and TIL (e.g., by allowing improper constructions into HTLC), (2) TIL and CLC without relying on the middle step in the form of HTLC, (3) HTLC and the modal logic TRIV

Logical rules:

$$\frac{}{\Gamma, x : T \vdash x : T}\ \textsc{hyp}$$

$$\frac{\Gamma, x : T \vdash s(x) : S}{\Gamma \vdash \lambda x.s(x) : T \to S}\ {\to}\mathrm{I} \qquad \frac{\Gamma \vdash r : T \to S \qquad \Gamma \vdash t : T}{\Gamma \vdash \mathsf{ap}(r, t) : S}\ {\to}\mathrm{E}$$

$$\frac{\Gamma \vdash t : T}{\Gamma \vdash \mathsf{triv}(t) : *T}\ {*}\mathrm{I} \qquad \frac{\Gamma \vdash r : *T \qquad \Gamma, x : T \vdash s(x) : S}{\Gamma \vdash \mathsf{exe}(r, x.s) : S}\ {*}\mathrm{GE}$$

Computation rules:

$$\mathsf{ap}(\lambda x.s, t) \Longrightarrow_{\beta} s(t)$$
$$\mathsf{exe}(\mathsf{triv}(t), x.s) \Longrightarrow_{\varepsilon} s(t)$$

Expansion rules:

$r \Longrightarrow_{E} \lambda x.\mathsf{ap}(r, x)$ where $r : T \to S$

$r \Longrightarrow_{E} \mathsf{triv}(\mathsf{exe}(r, x.x))$ where $r : *T$

Figure 1: Logical, computation, and expansion rules of HTLC

characterized by axiom $A \leftrightarrow \Box A$ (in HTLC, it is a logical truth that $T \leftrightarrow *T$), and between (4) the hyperintensional modality $*$ of HTLC and other related modalities, such as, e.g., the hyperintensional sentential modality *Hyp* considered by Sedlár (2021).

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An abstract Beth-like definability theorem

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Tarski (1930) laid down a series of axioms aiming to characterise a primitive notion of *consequence* and showed that, by means of this notion only, other metalogical concepts could be defined; among them the concepts of *theory*, *logical equivalence*, *consistency*, *completeness*.

Later, Tarski (1935) initiated a study of the metalogical concept of *definition* showing important analogies with the abstract approach taken in (Tarski, 1930). Even though the intimate connections of the later work with the previous one are emphasised by Tarski himself, he develops the study on definitions within a framework which is considerably less 'abstract' than that assumed to study the notion of consequence: The former presupposes sentences being endowed with an internal structure distinguishing variables and extra-logical terms, as the minimum setting for speaking about "the definability and the mutual independence of concepts" (Tarski, 1969, p. 296). By contrast, in the present paper my aim is that of establishing the fundamentals of an abstract theory of definitions in the same framework of (Tarski, 1930), by taking an arbitrary notion of consequence as the only primitive concept.

An abstract theory of definitions intends to capture analogue concepts studied by theories of definitions within formalised languages and logics. Therefore, we start by briefly recalling the fundamentals of the most developed and uncontroversial of such theories — the classical theory of definitions for first-order languages — restricting ourselves, for simplicity, to the case of the definition of a unary predicate in terms of a given language which only contains primitive non-logical symbols (no previously defined symbols).

Definitions in first-order logic

Let $\mathcal{L}$ be a first-order language with identity. The symbol $\vdash$ denotes the relation of (classical) logical consequence between sets of formulæ of $\mathcal{L}$ and formulæ of $\mathcal{L}$, equivalently defined, by the completeness theorem, either in terms of rules of inference or in terms of models.

We assume that among the non-logical constants of $\mathcal{L}$ there is a unary predicate $\mathbf{P}$ we want to define in terms of the other non-logical constants of $\mathcal{L}$. We denote by $\mathcal{L}^-$ the sublanguage of $\mathcal{L}$ built from the same non-logical constants of $\mathcal{L}$, except $\mathbf{P}$. For every set Φ of formulæ of $\mathcal{L}$ we denote by Φ^- the intersection $\Phi \cap \mathcal{L}^-$.

Let Σ be any set of sentences of $\mathcal{L}$. We understand the sentences which are in Σ but not in $\mathcal{L}^-$ as axioms added to the base theory Σ^- in order to define the predicate $\mathbf{P}$. The classical theory of definitions[1] has that the set of sentences Σ is a correct definition of $\mathbf{P}$ (in terms of the base theory Σ^-) iff Σ has both the following properties:

- *(Syntactic) Conservativeness*: Every sentence of $\mathcal{L}^-$ which is provable from Σ is already provable from Σ^-.
- *Eliminability* (or *explicit definability*): Every formula ϕ of $\mathcal{L}$ is provably equivalent modulo Σ to a formula ϕ^- of $\mathcal{L}^-$.

[1] See (Suppes, 1957) for a modern treatment.

The classical theory of definitions in first-order logic establishes that the two conditions of syntactic conservativeness and eliminability are jointly equivalent to the following model-theoretic condition: "Every model of Σ^- has *exactly* one expansion to a model of Σ". By removing from the latter property its existence claim, we obtain the following uniqueness condition on Σ which in literature is frequently called *implicit definability*: "Every model of Σ^- has *at most* one expansion to a model of Σ". The fact that implicit definability implies explicit definability is the statement of *Beth's definability theorem*.

Definitions in an abstract setting

We now turn to Tarski's abstract setting[2]. We work with an arbitrary non-empty set A and with a primitive notion of *consequence*, denoted by $\models$, between subsets and elements from A, which satisfies the following axioms:

- $\{x\} \models x$ (*reflexivity*).
- $X \subseteq X' \Rightarrow \forall x\,(X \models x \Rightarrow X' \models x)$ (*monotonicity*).
- $X \models Y \wedge Y \models x \Rightarrow X \models x$ (*transitivity*).

Following Tarski, we say that a subset X of A is *consistent* iff there exists $x \in A$ such that $X \not\models x$. We say that X is *maximal consistent* iff X is maximal with respect to inclusion in the family of all consistent subsets of A.

Definitions in first-order logic assume that the full object language $\mathcal{L}$ is split into two subsets: The set of the formulæ of $\mathcal{L}$ in which the distinguished predicate $\mathbf{P}$ occurs and the set of the formulæ in which $\mathbf{P}$ does not occur, the latter denoted by $\mathcal{L}^-$. Analogously, in the abstract setting we fix a distinguished subset A^- of A intended to play the role of "sub-language".

We can give the following abstract counterpart of corresponding notions involved in the classical theory of definitions. For X a subset of A, let $X^- = X \cap A^-$. Let $W \subseteq A$. We say that

- W is a *syntactic definition* iff W has the properties (with respect to A^-) of *non-creativity* (the same as syntactic conservativeness with $\models$ in the role of $\vdash$) and *abstract eliminability*, namely, every element $x \in A$ is $\models$-equivalent modulo W to an element x^- of A^-.
- W is a *semantic definition* iff for every maximal (in A^-) consistent subset X of A^- such that $W^- \subseteq X$, there exists one and only one maximal consistent set U such that $X \cup W \subseteq U$.

The above-mentioned abstract notions of definitions are motivated as follows. On the syntactic side, the properties of non-creativity and abstract eliminability are straightforward translations of the first-order notions of conservativeness and eliminability, obtained by defining the relevant notions in terms of the primitive consequence relation $\models$ in the role of the relation of first-order logical consequence $\vdash$. On the semantic side, the talk about models is replaced by talk about maximal consistent sets by observing that, in the first-order context, a set of formulæ (sentences) is maximal consistent if and only if is the set of all formulæ (sentences) which are satisfied (true) in a model endowed with an assignment to the variables of $\mathcal{L}$ of individuals taken from the domain of the model.

[2] See (Martin&Pollard, 2013) for a thorough exposition.

In order to establish the mutual relationships between the syntactic and semantic notions of definition, we need to specify further the relation of consequence and "sublanguage" we are dealing with. Let us assume that the relation $\models$ and the subset A^- satisfy the following properties:

1. $X \models x$ iff there exists a finite $Y \subseteq X$ such that $Y \models x$ (*Finiteness*).
2. For every $x \in A$ there exists $\overline{x} \in A$ which $\models$-contradicts x (*Classical Negation*).
3. For every $x, y \in A$ there exists $z \in A$ $\models$-equivalent to $\{x, y\}$ (*Binary Conjunction*).
4. The set A^- is closed under both Classical Negation and Binary Conjunction.

The above-mentioned properties allow us to prove the following

Thm 1. *Under the above assumption, a non-empty subset W of A is a syntactic definition iff is a semantic definition.*

The salient point in the proof of the above theorem is the following abstract version of Beth's definability theorem. Let us call *semantic determinability* the property: For any two maximal consistent sets U, U', if they both extends W and agree on A^-, then $U = U'$. Then, semantic determinability implies abstract eliminability. The proof of this latter claim exploits results obtained in (Dosen&Schroeder-Heister, 1988).

Abstract definitions and first-order logic

By interpreting $\models$ on the relation of first-order logical consequence *restricted to sentences* of $\mathcal{L}$, we obtain almost straightforwardly the following

Thm 2. *Let Σ be a set of sentences of $\mathcal{L}$. Then, the following are equivalent:*

1. *Any two expansions of elementarily equivalent models of Σ^- to models of Σ are elementarily equivalent.*
2. *Every* sentence *ϕ of $\mathcal{L}$ is provably equivalent modulo Σ to a* sentence *ϕ^- of $\mathcal{L}^-$.*

Analogously, by interpreting $\models$ on the relation of first-order logical consequence on *formulæ* of $\mathcal{L}$, we obtain

Thm 3. *Let Σ be a set of sentences of $\mathcal{L}$. Then, the following are equivalent:*

1. *Let $\mathcal{M}, \mathcal{M}'$ be models of Σ and let ν, ν' be assignments to the variables of $\mathcal{L}$ of individuals taken from the domains of $\mathcal{M}$ and $\mathcal{M}'$, respectively. If the reducts of $\mathcal{M}$ and $\mathcal{M}'$ to $\mathcal{L}^-$ satisfy (under the assignment ν and ν') the same formulæ of $\mathcal{L}^-$, then $\mathcal{M}$ and $\mathcal{M}'$ satisfy the same formulæ of $\mathcal{L}$.*
2. *Every* formula *ϕ of $\mathcal{L}$ is provably equivalent modulo Σ to a* formula *ϕ^- of $\mathcal{L}^-$.*

In the last theorem what is missed, in order to prove Beth's definability theorem as a corollary of its abstract version, is an independent proof that implicit definability implies the model-theoretic property (1).

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Truth, the correspondence platitude, and paradox

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Stephen Read (2008b) has argued for a new theory of truth formed by disambiguating an ambiguity in an accepted commonsense platitude about truth. Read advertises his theory as providing a particular solution to the Liar Paradox. The present work refutes an earlier objection to Read, reveals as flawed Read's claim that his theory produces the aforementioned solution to the Liar Paradox, identifies a further ambiguity in Read's theory as the cause of the flaw, and shows that an even more careful precisification of the commonsense platitude indeed produces the paradox-avoiding result Read claims his theory to produce.

It is well known that some instances of the *Disquotation Schema*

$$\text{`}P\text{' is true iff } P \qquad (DS)$$

lead to contradiction when combined with otherwise consistent theories. An archetypal example is the Liar Paradox: Consider the sentence

$$L \text{ is not true.} \qquad (L)$$

Suppose first that L is true. Since 'L is not true' is L, 'L is not true' is true, so—by DS—L is not true. Now suppose instead that L is not true. This time, by DS, 'L is not true' is true, whence L is true. Thus L is true iff L is not true, and, given that L is true or L is not true, L both is and is not true—contradiction.[1]

Crispin Wright (1992, pp. 25, 34) propounds that what underlies the plausibility of DS is the *Correspondence Platitude*

$$\text{`}P\text{' is true iff things are as `}P\text{' says they are,} \qquad (CP)$$

from which DS allegedly follows via the *Transparency Platitude*

$$\text{`}P\text{' says that } P. \qquad (TP)$$

It consequently appears that we must either (i) give up CP, (ii) give up TP, (iii) deny DS follows from CP and TP, (iv) impose other restrictions on the truth theory or the base theory, or (v) accept some contradictions. Giving up CP is costly, since it is to deny how the word 'true' is ordinarily understood, and—as Julian Dodd (2001, p. 75) points out—CP itself says so little that it is not disagreeable to correspondists, coherentists or deflationists. Giving up TP is also costly, since it likewise violates actual competent language usage.

Read (2008b) argues for option (iii). According to Read, DS may appear to follow from CP and TP, because CP is ambiguous. The ambiguity is between, on the one hand, setting the condition that things are in *some* way 'P' says they are, and, on the other, the condition that things are *however* 'P' says they are. Read (2008b, pp. 6–8) defends as a truth definition the presicification

$$\text{`}P\text{' is true iff things are however `}P\text{' says they are,} \qquad (T)$$

[1]The paradox can be formulated without appeal to *Excluded Middle*, but this is not relevant for the present discussion and would make for a needlessly longer example.

which he formalises as

$$\forall x(\mathrm{T}x \Leftrightarrow \forall p(x : p \rightarrow p)), \tag{T_f}$$

where '$x : p$' formalises 'x says that p', and '$\Leftrightarrow$' is the necessitation of the material biconditional, used by Read to formalise logical consequence.[2] Classical logic is assumed for the sentential connectives and individual quantifiers. Read further notes that even T alone is not precise enough: for a sentence to be true, not only is it required that things are however the sentence says they are, but things need to be however what the sentence says logically implies them to be. Hence, for the purpose of giving a definition of truth, Read closes what a sentence says under logical consequence and formalises the closure condition by

$$\forall x \forall p \forall q((p \Rightarrow q) \rightarrow (x : p \Rightarrow x : q)). \tag{C_f}$$

Read (2008b, pp. 11–13) then uses the Liar Paradox as a test case for the theory. He first argues that T_f and C_f have the consequence $\forall x(x : \neg\mathrm{T}x \rightarrow x : \mathrm{T}x)$. In particular, for a liar sentence L for which it holds that $L : \neg\mathrm{T}L$, the theory supposedly has the consequence that $L : \mathrm{T}L$. Read now judges that although the standard argument to the conclusion $\mathrm{T}L \rightarrow \neg\mathrm{T}L$ goes through, the second leg of the argument of the Liar Paradox is blocked, because unlike with DS, showing that L is not true no longer suffices to show that L is true: by T_f, to show that L is true requires us to show that things are however L says, and now L has been shown to say something besides $\neg\mathrm{T}L$, namely that $\mathrm{T}L$.

Greg Restall (2008a, pp. 231–232) uses Kripke semantics to define a logic that fits the above description, where the modality of '$\Rightarrow$' is $\mathbf{S5}$, and in which $L : \neg\mathrm{T}L \rightarrow L : \mathrm{T}L$ does not follow from T_f and C_f, refuting a part of Read's means to avoid the contradiction.[3] Restall (2008b, pp. 145–146) suggests Read's conclusion does not follow because Read's theory allows that it is possible for sentences to say something else than what they actually say. In this, Restall ignores a crucial feature of Read's (2008a, p. 210) theory: the theory is applied to interpreted sentences that have their content essentially. Read just neglects to formalise his premiss of necessity of content, which we formalise as

$$\forall x \forall p((x : p \rightarrow \Box x : p) \wedge (\neg x : p \rightarrow \Box \neg x : p)). \tag{N_f}$$

It is a simple matter to show that $L : \neg\mathrm{T}L \rightarrow L : \mathrm{T}L$ follows from T_f, C_f and N_f in Restall's logic. Indeed, Restall's logic is shown to be so impracticably strong for examining Read's formalised theory that in it $L : \neg\mathrm{T}L \rightarrow L : \mathrm{T}L$ follows from just N_f. To investigate further, we move to neighbourhood semantics to weaken Restall's logic by weakening the principles that govern ':' while retaining the principles that govern the quantifiers, the modal operators and the sentential connectives. It is shown that in the resulting logic $\{T_f, C_f, N_f\} \nvDash L : \neg\mathrm{T}L \rightarrow L : \mathrm{T}L$. Hence, even a faithful formalisation of Read's theory fails to produce his paradox-avoiding conclusion in the logics he claims the theory produces it.[4]

We claim the failure is a result of Read's presicification of CP falling short of the mark. To be exact, our claim is not that T is not a workable partial presicification of CP. Instead, we claim that C_f fails to capture what is intended. The formalisation C_f was intended to

[2]It is assumed that the individual quantifiers range over sentences. If not, T_f should be reformulated as $\forall x(\mathrm{T}x \Leftrightarrow \exists p(x : p) \wedge \forall p(x : p \rightarrow p))$ to prevent the theory from classifying non-sentences as true.

[3]Although adopting T_f or T and rejecting DS is enough to block the Liar, Read's approach gives an immediate explanation of why the Liar is not also true in addition to not being true, and is in that respect more attractive.

[4]A reviewer wondered whether there has been discussion of Read's proposal after 2008. There has, especially work where the theory is taken as an interpretation of the scholastic Thomas Bradwardine's solution to the Liar. For a very recent example of this, see (Read, 2025). However, none of the later discussion addresses Restall's counterexample or the related issues in any substantial way.

formalise, together with T_f, the observation that for a sentence to be true, things need to be however what the sentence says logically implies them to be. Unfortunately, the natural language phrase 'however what the sentence says logically implies them to be' may still mislead the unwary in a manner reminiscent of the ambiguity Read pointed out in CP. As it happens, what Read's T_f and C_f jointly formalise is that a sentence is true just in case things are in every way that *any* way-in-which-the-sentence-says-things-are logically implies them to be. But—crucially—if things are not in every way that things-being-in-*every*-way-that-the-sentence-says logically implies them to be, then the sentence cannot be true, given standard properties of logical implication. Thus, what ought to be formalised by the formal theory is that a sentence is true iff things are in every way that things-being-in-every-way-that-the-sentence-says logically implies them to be.[5] It is shown that it is not altogether straightforward to strengthen Read's closure principle C_f to capture this idea, but that in the context of T_f and N_f, the formalisation

$$\forall x \forall p((\forall q(x:q \to q) \Rightarrow p) \to x:p) \tag{C'_f}$$

is equal to the task.

Finally, it is shown that the achieved precisification can be viewed as the general rule from which DS follows in the case of *strongly transparent* sentences that in a sense say precisely what they appear to say. We tentatively propose, then, that the new theory of truth is not in opposition to the strong intuitions in favour of DS, but lends the intuitions support in the cases where they are creditable in the first place, and reins in the intuitions—and DS—, where they tend to go too far, like in the case of the Liar. The further merits of the presicified theory—or lack thereof—will be investigated in a later series of articles, which will include proof of the theory's extraordinary resilience to paradoxes that fit Priest's (2002, pp. 133–136, 276–280) *Inclosure Schema*, which arguably captures the collection of paradoxes of self-reference.[6]

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[5]In other words, T_f and C_f jointly formalise the claim that a sentence is true just in case things are however their being *anyhow* the sentence says logically implies them to be. Nevertheless, if things are not however things-being-*however*-the-sentence-says logically implies them to be, then the sentence cannot be true.

[6]A reviewer has commented that the idea that sentences say multiple things is counterintuitive. There is a possibility for reconciliation between the theory and the intuitions: The theory has the resources to define a stricter notion of saying for which sentences say only one thing up to logical equivalence. Let $t \,\underline{:}\, \varphi =_{\text{def.}} t:\varphi \land \forall p(t:p \to (\varphi \Rightarrow p))$. The theory then proves $\exists p(x \,\underline{:}\, p)$ for all sentences x, and $\forall x \forall p \forall q(x \,\underline{:}\, p \land x \,\underline{:}\, q \to (p \Leftrightarrow q))$.

Towards the semantics of subDL

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Weber (Weber, 2021) has proposed an ambitious project in the reformulation of mathematics in a paraconsistent framework. To that end, he and Badia proposed the propositional logic **subDL** and its first order extension **subDLQ** (Badia,Weber, 2019). Like any such logic, **subDL** faces the problem of reconciling sufficient strength to capture classical theories with sufficient weakness to permit radically non-classical ones. Since naïve set theory is one of the latter, the logic needs in particular to avoid the structural rule of contraction. The chosen approach is to make use of *two* implication connectives: one satisfies the postulates of the weak relevant logic **DW** (Brady, 2006) including full contraposition and replacement of equivalents, while the other satisfies weakening and in fact all postulates of the strong substructural logic **BCK**. Conjunction and disjunction, which I shall write as $\otimes$ and $\oplus$ respectively, are treated in a unique way. At the **DW** level, they are associative and commutative, and each other's De Morgan duals, while at the **BCK** level $\otimes$ is multiplicative conjunction ("fusion" or "tensor product") while $\oplus$ is additive disjunction. Despite the apparent tension between these readings, algebraic models of **subDL** do exist, and show that the logic is indeed paraconsistent and does not validate contraction for either of the implications.

There is, however, no known coherent semantic or proof-theoretic account of the logic beyond "whatever structures make the axioms hold" and this limits its applicability for the purpose of mathematics, since mathematical theories without models can hardly constitute a viable alternative to the standard brands. The present paper is intended to help put the entire project on a firmer footing by suggesting at least one way to secure a logic in the style of **subDL** but with clear and well-understood semantics.

We begin with an axiomatisation. The language of **subDL** has primitive connectives $\bot$, $\neg$, $\otimes$, $\rightarrow$, $\Rightarrow$ with the standard formation rules. There are defined connectives: $A \oplus B$ is defined as $\neg(\neg A \otimes \neg B)$, and there are biconditionals $A \leftrightarrow B$ defined as $(A \rightarrow B) \otimes (B \rightarrow A)$, and $A \Leftrightarrow B$ as $(A \Rightarrow B) \otimes (B \Rightarrow A)$. As axiom schemata we may take:

a1. $A \rightarrow A$
a2. $\bot \rightarrow A$
a3. $\neg\neg A \rightarrow A$
a4. $(A \rightarrow \neg A) \rightarrow (B \rightarrow \neg A)$
a5. $((A \rightarrow B) \otimes (B \rightarrow C)) \rightarrow (A \rightarrow C)$
a6. $\neg(A \otimes \neg A)$
a7. $(A \otimes B) \rightarrow A$
a8. $(A \otimes B) \rightarrow (B \otimes A)$
a9. $((A \otimes B) \otimes C) \rightarrow (A \otimes (B \otimes C))$
a10. $(A \rightarrow B) \Rightarrow (A \Rightarrow B)$
a11. $\neg(A \Rightarrow B) \Rightarrow \neg(A \rightarrow B)$
a12. $((A \otimes B) \Rightarrow C) \Leftrightarrow (A \Rightarrow (B \Rightarrow C))$
a13. $((A \Rightarrow C) \otimes (B \Rightarrow C)) \Rightarrow ((A \oplus B) \Rightarrow C)$
a14. $A \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B))$

To these are applied the rules of detachment, transitivity, replacement and affixing:

$$\frac{A \Rightarrow B \qquad A}{B} \qquad\qquad \frac{A \Rightarrow B \qquad B \Rightarrow C}{A \Rightarrow C}$$

$$\frac{A \leftrightarrow B \qquad C(A)}{C(B)} \qquad\qquad \frac{A \to B \qquad C \to D}{(B \to C) \to (A \to D)}$$

Detachment and transitivity for the single arrow are easily derived, as is affixing for the double one, and distribution of $\otimes$ over $\oplus$ in the form

$$(A \otimes (B \oplus C)) \Leftrightarrow ((A \otimes B) \oplus (A \otimes C))$$

Distribution of $\oplus$ over $\otimes$, however, is not derivable, and is not wanted as it leads directly to contraction.[1]

While **subDL** itself remains puzzling, there are broadly similar ideas in the relevant logical tradition that are much better understood. One is the concept of *enthymematic implication*, introduced by Anderson and Belnap (Anderson,Belnap, 1961) which was investigated at the time by Meyer and others (Meyer, 1970; Meyer-Giambrone, 1981) and continues to be of interest (??gaard, 2020).

Let the underlying logic **DW** be formulated not only with $\to$ and $\neg$ but also with additive (distributive lattice) connectives $\wedge$ and $\vee$, where $A \vee B$ is defined as $\neg(\neg A \wedge \neg B)$. Its extension $\mathbf{DW}^{\circ t}$ adds fusion $\circ$ and the sentential constant t. In addition to axioms a1–a4 and the rules of detachment (for the single arrow) and affixing, $\mathbf{DW}^{\circ t}$ has axioms

a14. $(A \wedge B) \to A$
a15. $(A \wedge B) \to B$
a16. $((A \wedge B) \wedge (A \to C)) \to (A \to (B \wedge C))$
a17. $(A \wedge (B \vee C)) \to ((A \wedge B) \vee C)$

along with the rules:

$$\frac{A \qquad B}{A \wedge B} \qquad \frac{A}{t \to A} \qquad \frac{t \to A}{A} \qquad \frac{A \to (B \to C)}{(A \circ B) \to C} \qquad \frac{(A \circ B) \to C}{A \to (B \to C)}$$

DW has the same axioms but only the first of these five rules. In $\mathbf{DW}^{\circ t}$ we may *define* a double arrow: let $A \Rightarrow B$ be $(t \wedge A) \to B$. This connective satisfies wekening in that if B is a theorem, so is $A \Rightarrow B$. It does not generally satisfy the B or C axioms of **BCK**, but it does represent a step towards a **subDL**-like logic. Writing the conjunction $t \wedge A$ as $\Box A$, note the theorems:

$\Box A \to A$
$\Box(A \to B) \to (\Box A \to \Box B)$
$\Box A \to \Box\Box A$
$\Box(A \wedge B) \leftrightarrow (\Box A \wedge \Box B)$

Moreover, if A is a theorem, so is $\Box A$. So the $\Box$ operator yields a normal modal logic with the flavour of **S4**, albeit with a substructural logical base. To be sure, there are a few decidedly non-**S4** properties, like the equvalence between $\Box(A \vee B)$ and $\Box A \vee \Box B$, but the overall similarity to a known body of logical theory is suggestive and encouraging.

[1]The original formulation (Badia,Weber, 2019) had distribution as an axiom, with the single arrow in place of the double one. This was a mistake (Slaney, 2025), corrected for present purposes.

There is more: with this definition of $\Rightarrow$, we also obtain the corresponding intensional conjunction: $A \otimes B$ may be defined as $\Box A \circ \Box B$. Modelling conditions for $\Rightarrow$ and $\otimes$ in Routley-Meyer frames (Routley-Meyer, 1973) are easily derived from those for the **DW**$^{\circ t}$ connectives. In any model $\mathscr{M}$ with normal worlds N:

$w \models_{\mathscr{M}} A \Rightarrow B$ iff for all $x \in N$ and all y such that $Rwxy$, if $x \models_{\mathscr{M}} A$, then $y \models_{\mathscr{M}} B$
$w \models_{\mathscr{M}} A \otimes B$ iff for some x, y in N, $Rxyw$ and $x \models_{\mathscr{M}} A$, and $y \models_{\mathscr{M}} B$

To obtain all of **BCK**, for the double arrow, postulates making $\otimes$ associative and commutative need to be imposed just on *normal* worlds: for all a and b in N and for all x, if $Rabx$ then $Rbax$; moreover, for a, b, c in N and all d, if there is x such that $Rabx$ and $Rxcd$ then there is y such that $Rbcy$ and $Rayd$.

It is convenient to have a name for the resulting logic. Let **DWe** be **DW**$^{\circ t}$ with the suggested enthymematic definitions of $\Rightarrow$ and $\otimes$ and with the addition of axioms a8 and a9. **DWe** is fairly close to **subDL**, but the two logics are not identical. Axioms a5, a13 and a14 are not true of enthymematic implication. Crucially, a13 cannot be imposed without collapsing **DWe** to classical logic. Proof of this assertion is omitted from the present abstract for reasons of space. Conversely,

$$((A \otimes B) \Rightarrow C) \to ((A \otimes B) \to C)$$

for example is a theorem scheme of **DWe** but not of **subDL**. It is not hard to show that contraction is not an admissible rule in either of the two logics.

A question arising immediately is whether the addition of enthymematic implication with **BCK** axioms conservatively extends **DW**. The answer is yes: a simple construction (again omitted on grpunds of space) shows that an arbitrary Routley-Meyer frame for **DW** may be embedded in one satisfying the additional constraints on normal worlds. Conservative extension also holds in the opposite direction: the negation-free part of distributive **BCK** (Ono-Komori, 1985) may be represented as the enthymematic fragment of **DWe**.

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Universal judgments and Kant

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The aim of this paper is generally to find out some philosophical assumptions of classical modern logic, especially in comparison with older Aristotelian logic. In particular, I focus on the problem of reinterpreting the general affirmative judgments, one of the traditional and essential components of Aristotelian logic. It is well known that if we rewrite the judgments of the square of opposition in the language of first-order logic, then most of the relations within the square collapse, except the relation of contradiction. There are several possible explanations for why this is the case. The key one is that while in Aristotelian logic general judgments in the square of opposition are understood as categorical subject-predicate judgments (SaP), in modern logic they are understood as expressing conditional statement: ($\forall x(S(x) \rightarrow P(x))$), i.e., in the older terminology, they are hypothetical judgments. The relations within the square of opposition collapse because the original square of opposition is conceived as a square of opposition of categorical, not hypothetical judgments.

Now, however, the question is why the general judgments have been interpreted this way. Much has been said about mathematical reasons, but little has been said about philosophical reasons. My thesis is that the acceptance of the hypothetical interpretation of general judgments was because it reflected, on a logical level, contemporary efforts to re-found noetic and to overcome psychologism, which was prima facie a noetic problem.

I will look for evidence of this thesis in Russell's famous article *On Denoting*. In the first part of the article, Russell briefly states his theory of denoting phrases. After explaining the most basic denoting phrases, he moves on to categorical subject-predicate judgments. And here he writes:

> "Consider next the proposition 'all men are mortal'. This proposition (as has been ably argued in Mr. Bradley's Logic, Book I, Chap. II.) is hypothetical and states that if anything is a man, it is mortal. That is, it states that if x is a man, x is mortal, whatever x may be." (Russell 1905, 481)

The text in brackets is a footnote in Russell's article. Russell is referring to the arguments of a leading British Idealist F. H. Bradley. The assumption of a consensus of two such philosophically disparate authors is weak, and the found agreement is therefore rather surprising.

Bradley's claim about the hypothetical nature of categorical judgments is based on a specific conception of the content of a judgment. "Ideal content," as Bradley calls it, is ontologically a somewhat special entity in his conception, but one that must be understood against the background of Bradley's polemic with the early British empiricists. The full dispute is all about noetic, and it's about Bradley wanting to escape the sceptical implications of the traditional theory of judgment in conjunction with the consistently sensationalist position of the early British empiricists. If a judgment is a conjunction of ideas, and ideas as the only "material" of human knowledge are nothing other than the contents or facts of the mind, then it is difficult to resist the fatal closeness of the noetic subject to extramental reality. Bradley's solution is based on two steps: a) a different conception of the term "idea", which is deprived of its close interdependence with the mind in which it occurs. But the ontological status of ideas so conceived, which are "cut off from existence" still does not provide the required contact with extramental reality. Therefore, b) understood in this way content of the judgment is as a whole predicated on reality.

Russell, after his break with Hegelianism, certainly disagreed with the whole series of beliefs that Bradley held. Nevertheless, the pursuit of a new, more certain foundation of noetic and the struggle against the belief that the noetic subject has access only to the contents of his mind (psychologism!) are crucial points of agreement that make Russell agree with Bradley's above revision of the theory of judgement. For both of them, the content of judgment is a certain abstract thing attributed to some unspecified reality. As Bradley puts it – "the actual judgement asserts that $S{-}P$ is forced on our minds by reality x. And this reality, whatever it may be, is the subject of the judgment" (Bradley 1912, 40-41). But Bradley himself writes that the thesis of the hypotheticality of categorical judgments occurred to him while reading Kant's follower Herbart (Bradley 1912, 42). For the origins of this conception, then, we need to focus on Kant's philosophy. Is it possible to find any place in Kant where he talks about different conceptions of judgment? I think it is specifically his distinction between analytic and synthetic judgments.

According to this division, in all $S{-}P$ judgments, of the form "A is B," the predicate belongs to the subject in one of two ways: either as something that is in the concept of A (covertly) contained (analytic judgment); or "B lies entirely outside the concept of A, although it is associated with this concept" (synthetic judgment). Thus, in the synthetic judgment, the question immediately arises as to what causes the true predication B of A when B "lies entirely outside" A. Kant considers the example of synthetic judgment in the sentence "All bodies are heavy" and claims of it:

> "For synthetic judgments, I must have something else in addition to the notion of subject (X), on which the reasoning relies, in order that a certain predicate, which in that notion . . . does not lie in that concept, yet it recognizes it as belonging to it. . . . when I look back at the experience from which I abstracted this notion of body, I find that . . . there is always a heaviness connected with the aforementioned features. Experience is therefore that X which lies outside the concept of A and on which the possibility of synthesis before the predicate of gravity B with the concept of A."

Incidentally, then, we encounter again that tentative hint of the use of the variable x that we saw above in Bradley. According to the German philosopher W. Cramer, the structure of such a judgment could be schematized as follows: *That (X) which I recognize under the determinant A, I recognize also under determination B*. From this schema, it is even more evident what the actual interpretation of Kant was aiming at: if concepts are essentially predicative, then (in synthetic judgment) in the act of judging, both concepts are predicated on something in common. This commonality only makes possible the union of the predicate with the subject. From here it is only a step to take the common as the real subject of the judgment and to modify the structure of the judgment so that if the common belongs under A, then it also belongs under B, i.e. to understand this judgment as conditional, hypothetical. But the absolutely crucial question for Kant now is what mediate between the phenomenon and the concepts in either of the two predications, especially since for Kant the phenomenon and concept are the result of the activity of a completely different cognitive power. Clearly, there must be a third thing that must be on one side homogeneous with the categories, and with the phenomenon on the other, and which allows for the application of the category to the phenomenon. This mediating idea must be pure (without anything empirical), and yet intellectual on the one hand, and on the other hand, sensuous. Such an idea is the transcendental schema. Needless to say, then, how great a role in this Kant's (completely new!) theory of predication, the transcendental schema plays. But it is at this point, as Kant's interpreters agree, that Kant fails to come up with a clear, comprehensible coherent explanation of this key concept.

I believe that these protracted problems with the concept of transcendental schema causes that Kant's original theory of prediction did not reach a clear application in logic, although Kant's noetics otherwise marked an absolutely fundamental breakthrough. It was only Frege, who ingeniously transferred from contemporary mathematical practice the notion of function, which closed the problematic gap in Kant's theory.

If my previous interpretation was correct, then in addition to purely mathematical motivations, the success of this concept is connected with certain philosophical (specifically noetic) problems and even more with certain proposals for solutions to those problems. In consequence, this is about Kant's distinction between two sources of knowledge and, by analogy, between two kinds of judgments, analytical and synthetic. These distinctions, especially the latter, should therefore be fundamental. And that is why it is most remarkable that in the second half of the twentieth century in analytic philosophy (but not only there!), a powerful critique of these initial noetic assumptions revels. It is in particular Quine's famous critique of the distinction between analytic and synthetic judgments and Davidson's critique of the so-called third dogma of empiricism, the distinction between schemata and non-schematized content. The question is whether this criticism can shake the modern interpretation of general judgments.

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Petr Vopěnka, some early results in set theory

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Petr Vopěnka (born 1935, died 2015) was an important figure in Czech mathematics and in Czech academia in general. As a student he was influenced by Eduard Čech and by Ladislav Svante Rieger. In the beginning of his professional carrier, he left visible traces in classical set theory. During the 1960s he ran a seminar where not only new results were obtained but also several young people became respected researchers. Indeed, he was a stimulating teacher and an excellent speaker, and running a seminar in his concept meant getting the participants (like Bohuslav Balcar, Leo Bukovský, Petr Hájek, Karel Hrbáček, Tomáš Jech, Karel Příkrý, Antonín Sochor, Petr Štěpánek) to cooperate. Petr Hájek considered Vopěnka his unofficial thesis advisor. Objectives of this group were turning Cohen's theorem about the continuum hypothesis into a method that would yield further independence results, or reworking some model constructions as (syntactic) interpretations. Their Boolean-valued models (and ∇-models) are in fact a variant of the method of *forcing*.

Later during the 1970s Vopěnka (probably in cooperation with Petr Hájek) developed the Theory of Semisets TS, and then he proposed the Alternative Set Theory AST. Here is a brief explanation of these two axiomatic theories. In a theory where the primary concept is *class*, a *set* is defined as a class that is an element of some class, and a *semiset* is defined as a subclass of a set. TS was invented in connection with proving and explaining independence results. Proper semisets (semisets that are not sets) may or may not exist in it, and Gödel-Bernays set theory GB (now often referred to as NBG) can be obtained by adding the axiom "every semiset is a set" to TS. Then in AST, proper semisets positively exist and play a role, but there is no cardinal arithmetic. The main motivation to study AST is to reconstruct infinitesimals, a concept that once existed in classical mathematical analysis.

Yet later Vopěnka worked in the history of mathematics and in its philosophy. He was interested in the nature of the collection of all natural numbers and in the concept of infinity. He also considered theological motivations in mathematics.

Some people disagree with Vopěnka's philosophical ideas, or consider AST a dead end in mathematical research. And there may be something to their views. Nevertheless, Vopěnka's influence was considerable and his work is undoubtedly an interesting field of historical research.

This talk, which is a part of a project in history and philosophy of mathematics, deals with Vopěnka's classical set theory period, and in fact with only one aspect of it, *definable cuts in* GB. We thus omit independence results and the birth of forcing. The relationship between Boolean-valued models, ∇-models and possibly intuitionistic logic could be a subject of further investigation.

In an axiomatic theory where natural numbers are available, i.e. in a theory T in which some arithmetic is interpretable, a formula $\varphi(x)$ is a *(definable) cut* if the three sentences $\forall x(\varphi(x) \rightarrow (x \text{ is a natural number}))$, $\varphi(0)$ and $\forall x(\varphi(x) \rightarrow \varphi(x+1))$ are provable in T. A cut $\varphi(x)$ is *proper* if $T \nvdash \forall x((x \text{ is a natural number}) \rightarrow \varphi(x))$. A simple example is as follows. Since $\forall x(x+0=x)$ is an axiom of Robinson's arithmetic Q and the sentences $0+0=0$ and $\forall x(0+x=x \rightarrow 0+(x+1)=x+1)$ are provable in Q, the formula $0+x=x$ is a cut in Q. As Q $\nvdash \forall x(0+x=x)$, this cut is proper. Clearly, a proper cut in T demonstrates that the full induction principle is not provable in T. A cut can be altered to a "shorter" cut that is closed under addition and multiplication, and possibly satisfies some additional property of numbers expressed by a sentence α. In Q the sentence α can be, for example, the associativity and commutativity of operations, and the whole reasoning then shows that Q interprets $(\mathsf{Q}+\alpha)$.

Definable cuts are used in several important papers, one example being [Pud85]. Before Pudlák, Solovay used cuts in GB to construct a set sentence α such that GB interprets $(\mathsf{GB}+\alpha)$, but ZF does not interpret $(\mathsf{ZF}+\alpha)$. This is an unpublished but widely cited result contained in [Sol76]. Yet before [Sol76], the mere fact that GB admits proper cuts (without discussing interpretations) was included in [VH73].

In this talk we will claim, and provide an evidence for it, that Vopěnka was the first to notice that proper cuts in GB exist. We will consider the relevance of Vopěnka's habilitation thesis [Vop64] to this claim. We will try to put [Vop64] in the context of research that took place in the 1960s.

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A topical approach to the logic of fiction

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The logic of fiction (addressed at questions of what inferences we can draw *about* a fiction from claims made *within* the fiction) has been a long studied, though less visible, area in philosophical logic. (Woods , 1974) is monograph is an early example of the study of the use of non-classical logics in answering such issues, and a more recent paper by Proudfoot (Proudfoot , 2018) has argued that there is no logic of fiction. Still others (Berto and Jago , 2019) have proposed the use of impossible/open worlds to address issues in the field.

In this talk, we propose an alternate account which takes *topics* to be an essential ingredient. That is, we take those propositions to follow from a story to be those which (1) follow *tout court* and (2) are *on topic* w.r.t. the story. In the theory of topics promulgated in (Tedder , Forthcoming), we can model the topics on a space of propositions (structured as an algebra) by appeal to the subalgebra lattice of that algebra. Note that in such a setting we can define joins as:

$$\bigsqcup_{i\in I}\mathscr{B}_i = \mathscr{A}\big[\bigcup_{i\in I}\mathscr{B}_i\big]$$

So a natural way to proceed to model the consequences of a fiction, modeled by a set of propositions from the algebra, is to take the filter generated by that set *in the subalgebra generated by that set*. This is just the intersection of the filter generated by that set and the subalgebra generated thereby, and captures nicely the proposal of requiring points (1) and (2) mentioned above.

Fix an ordered algebra $\langle \mathscr{A}, Op^{\mathscr{A}}, \leq\rangle$, understood as a model of propositions. Given $\Gamma \subseteq \mathscr{A}$, let:

1. $[\Gamma)_{\mathscr{A}}$ be the filter generated by Γ, i.e.:

$$[\Gamma)_{\mathscr{A}} = \{y \in \mathscr{A} \mid \exists_{i\in I} x_i \in \Gamma(\bigwedge_{i\in I} x_i \leq y)\}$$

2. $\mathscr{A}[\Gamma]$ be the subalgebra of $\mathscr{A}$ generated by Γ, i.e. the least subalgebra $\langle \mathscr{B}, Op^{\mathscr{B}}, \leq\rangle$ of $\mathscr{A}$ such that $\Gamma \subseteq \mathscr{B}$

Then the proposed interpretation of "p follows from story Γ" (i.e., Γ is the set of propositions we take the story to express, and p the proposition expressed by some putative entailment) is:

$$p \in [\Gamma)_{\bigsqcup_{q\in\Gamma}\mathscr{A}[\{q\}]} \iff \exists_{i\in I} x_i \in \Gamma(\bigwedge_{i\in I} x_i \leq p) \;\;\&\;\; p \in \bigsqcup_{q\in\Gamma}\mathscr{A}[\{q\}]$$

We can adapt this definition to fix a relation on formulas of some language as usual, i.e. by taking the class of homomorphisms from the language to the target algebra, and thus define the obvious relation, taking $v\Gamma = \{v\psi \mid \psi \in \Gamma\}$ and $var\psi$ to be the set of atomic subformulas of ψ:

$$\Gamma \Vdash_{\mathscr{A}} \varphi \iff \forall v \in Hom(Fm, \mathscr{A})(v\varphi \in [\Gamma)_{\bigsqcup_{\psi\in\Gamma}\bigsqcup_{q\in var\psi}\mathscr{A}[\{vq\}]})$$

and generalise this to concern classes of algebras, as per usual.

We argue that this relation is a solid candidate for capturing the target notion in the logic of fiction, and furthermore that it has some desirable properties in common with logics of *analytic implication*. In particular, we'll assess whether this proposal resolves the significant issues in the logic of fiction as well as, or better, than the impossible/open worlds approach. In doing so, we draw comparisons to recent work in the logic of imagination such as (Badura , 2022).

In addition, we'll see how consequences in the logic of fiction reflect back on the subalgebra modeling of topics. For example, thinking in terms of stories provides a natural answer to the question of why the join of topics $\bigsqcup_{i\in I}\mathscr{B}_i$ in an algebra $\mathscr{A}$ should potentially be such that:

$$\bigcup_{i\in I}\mathscr{B}_i \subsetneq \bigsqcup_{i\in I}\mathscr{B}_i$$

A case where we should expect this to happen is in the case of *shared universes*. If we combine multiple stories, we may invoke new topics which none of the component stories themselves concern. For example, when we combine stories involving *magic* and *quantum mechanics*, such as some Marvel comics and movies do in the combination of the characters Dr. Strange (a magic user) and Ant Man (a scientist who can shrink to travel among quanta), questions are raised such as "What are the magical properties of quanta?", neither of the component stories concern. Related issues can be invoked to explain the potential failures of distributivity for $\bigcap$ and $\bigsqcup$.

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The logic of object relations

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This paper contributes to the formalisation of unconscious thinking by developing tools for the logical analysis of mental operations involving internal objects—background figures of the person's unconscious. The paper is based on the psychoanalytic theory of object relations and assumes that the operation of the unconscious mind adheres to its own logical standards. Our account of intentionality draws on Graham Priest's logic of intentionality (Priest, 2005) and Linda Brakel's analysis of unconscious attitudes (Brakel, 2009).

The 'logic of the unconscious'. The hypothesis that unconscious thinking has an inherent logically grounded rationality derives from Freud's insights into the characteristics of primary process cognition (Freud, 1953). Silvano Arieti expanded on Freud's ideas through clinical observations, exploring "paleological thinking" as primordial cognition manifesting in dreams, mental illness, creativity, and wit (Arieti, 1976). According to Arieti, the key feature of the primary process is the principle of "identification upon similarity," previously proposed by E. Von Domarus. According to this principle, the primary process tends to identify two different objects based on their subjectively chosen common attribute (a distinctive property).

In the second half of the 20th century, psychoanalysis underwent a major shift in its understanding of the unconscious. Freud's structural model of the relationships between the id, ego and superego was replaced by "a model of an inner world structured by phantasied internal object relationships" (Ogden, 2011, 926). An '(internal) object relation' is a psychological structure derived from early interactional experiences with significant others, such as parents or caregivers (Auchincloss, Samberg eds., 2012, 175-178). From the standpoint of object relations theory, much of the mind's unconscious activity is rooted in its attempts to project internal object relations outwards and reproduce them in interpersonal relationships. However, studies of unconscious thinking have not yet integrated the object relations perspective.

Example of object relations. Marvel is a man who once lived as a young boy on his father's neglected farm in Illinois. His father loved Marvel and controlled him at the same time. Today Marvel is over 40 and lonely. He sees more and more of his father's features in the mirror. Marvel's wishful phantasy is that one day he will have someone to care for. One day, Marvel goes to the pet shop and buys a smooth-haired hamster. The hamster has chubby cheeks—just like little Marvel in his photos. He also buys a cage for the hamster. Here, the hamster is Marvel's external object, while his internal objects are 'Father' and 'Little Marvel'. The chubby cheeks are a distinctive property based on which Marvel projects 'Little Marvel' onto the hamster. Psychoanalytically speaking, Marvel is transferring 'Little Marvel' onto the hamster and displacing his feelings from 'Little Marvel' to the hamster.

Logic of Projective Intentionality (LPI). The term 'projective intentionality' refers to the significance of phantasy-based intrapsychic content in mental acts. LPI is a multimodal logic based on Priest's (2005) logic of intentionality.

Consider a first-order language with constants $a, b, c, \ldots$, variables $z, y, x, \ldots$, n-place predicates $P_n, Q_n, R_n, \ldots$ (for all $n \geq 1$), including the unary predicate of existence $\mathcal{E}$ (Priest, 2005, 14), an equality $=$ and a versification predicate $\rightsquigarrow$ ($x \rightsquigarrow y$ is read as 'y is a version of x'), connectives $\wedge, \vee, \sim, \supset, \equiv$, quantifiers $\exists$ and $\forall$, a 'causal necessity' operator $\Box$, a 'causal implication' $\mapsto$, and the propositional constants $\top$ and $\bot$. We assume the division of constants and variables into types, such as the names of persons, internal objects, and external objects.

We also introduce intentional operators, in particular the operators of phantasy Φ and subjective perception Π. If p is a person and φ is a formula, $\Phi_p\varphi$ means 'p phantasies that φ'. Phantasy is the basic unconscious cognitive attitude (Brakel, 2009). Subjective perception is an overall cognitive attitude introduced on the presupposition that each person perceives things in her own way. It involves a reality check and may include fragments of phantasies, imagination, knowledge, and belief. For any person p and any formula φ, we write $\Pi_p\varphi$ for 'p subjectively perceives that φ'.

We define the notions of term and formula in a standard way with one exception: under the scope of intentional operators, no occurrences of $\Box$ or $\mapsto$ are allowed.

A PI-model is a structure $\langle D, B, O, R, I, s_0, T, \delta\rangle$, where D is a domain of objects, B is a set of persons (bearers of intentional states, $B \subseteq D$), O is a set of internal objects ($O \subseteq D \setminus B$), R is a set of 'real situations', I is a set of 'intentional situations' such that $R \cap I$ is $\emptyset$, $T \subseteq R \times R$ is a 'triggering relation', $s_0 \in R$ is the 'current' situation, and δ is a denotation function.

Intuitively, real situations connected by the relation T model the stages of a psychological process involving unconscious thinking: $T(r, u)$ (we will also say in this case that u is a *resulting situation* regarding r) means that u contains all causal consequences of r. Such modelling of causality is achieved by defining the semantic conditions for causal implication in a way that implements the intuition of the causal law: for any real situation s and any evaluation ν, $s \Vdash_\nu \varphi \mapsto \psi$ iff for all $r, u \in R$ such that $T(r, u)$, if $r \Vdash_\nu \varphi$ then $u \Vdash_\nu \psi$.

The denotation function is defined as in (Priest, 2005, 9-11), but we require it to conform to classical logic. Thus, if Ψ is an intentional operator, $\delta(\Psi)$ is a function that maps each $b \in B$ to a relation $C^b_\Psi \subseteq (R \cup U) \times I$. C^b_Ψ is the relation of intentional complementarity: intuitively, $C^b_\Psi(s, w)$ means that w realises all Ψ-attitudes of b in s. The relation T is required to be dense and extendable, and so is the relation C^b_Ψ for every b and Ψ. The denotation of $=$, defined as a subset of $D \times D$, in intentional situations may differ from its standard value in real situations, while remaining reflexive, transitive, and symmetric. Persons and external objects can only exist in real situations, while internal objects can only exist in intentional situations. The denotation of $\rightsquigarrow$ is a subset of $O \times (D \setminus O)$; this relation helps to model projective perception in which an internal object is transferred onto a real one.

The validity of a formula on a model is defined as truth under all evaluations in all real situations. The resulting semantics is that of constant domain: all objects are available in all worlds, although some objects may not exist in some worlds. Among the axioms justified by this semantics are the substitution axiom of identicals in intentional contexts $\forall u \forall v(\Psi(u = v) \supset (\Psi\varphi(u) \supset \Psi\varphi(v)))$, and those of density, extendability, and distribution over implication, for causal necessity as well as for both intentional operators.

The properties of internal objects. To make our assumptions about the universal properties of internal objects, we will need a second-order extension of our logic. In addition to predicate constants, we introduce predicate variables. We will use U as a property variable and Q as a two-place predicate variable.

Suppose that p has an internal object o, p and o being constants of the required sorts.

(P1) Existence-in-phantasy: $Exi_p(o) \equiv_{def} \Phi_p\mathcal{E}(o)$.

(P2) Persistence of phantasies: $Per_p(o) \equiv_{def} \forall Q(\Phi_p Q(p, o) \mapsto \Phi_p Q(p, o))$.

Now we can formally analyse the notion of distinctive property. According to our approach, distinctive properties serve two purposes: (1) to identify an external spatiotemporal object with the internal object in phantasy and (2) to identify it as a version (or a 'projection') of the internal object at the level of subjective perception. That is, for any distinctive property

M of the object, the second-order predicate $Dis^o_p(M)$ must hold: $Dis^o_p(M) \equiv_{def} \Phi_p M(o) \wedge \forall z(\mathcal{E}(z) \wedge \Pi_p M(z) \mapsto \Phi_p(o = z) \wedge \Pi_p(o \rightsquigarrow z))$.

(P3) Externalisability: $Ext_p(o) \equiv_{def} \exists U(Dis^o_p(U))$.

To sum up, internal objects are existing-in-phantasy parties to persistently phantasied internal object relations, externalisable by virtue of their distinctive properties.

Logical analysis of object relations. Consider the predicate $Obj^U_x(y)$, 'y is an internal object of the person x with a distinctive property U'. The following axiom (A1) of the internal object sums up our assumptions and justifies the following theorems (T1)-(T3):

(A1) $\forall U \forall x \forall y(Obj^U_x(y) \equiv \mathcal{E}(x) \wedge Exi_x(y) \wedge Per_x(y) \wedge Dis^y_x(U))$.

(T1) Transference: $\forall U \forall x \forall y \forall z(Obj^U_x(y) \wedge \mathcal{E}(z) \wedge \Pi_x U(z) \mapsto \Phi_x(y = z) \wedge \Pi_x(y \rightsquigarrow z))$.

The theorem states that the perception of a distinctive property in an external object leads to externalisation at the levels of phantasy and subjective perception.

(T2) Externalisation: $\forall Q \forall x \forall y \forall z(\exists U(Obj^U_x(y) \wedge \Phi_x Q(x, y) \wedge \mathcal{E}(z) \wedge \Phi_x(y = z) \mapsto \Phi_x Q(x, z))$.

In the situation of transference, any relation with an internal object is externalised in phantasy.

Some phantasy-based object relations (including object-attitudes) tend to expand beyond the limits of phantasy alone. Such relations can be defined by means of a property $E(Q)$, 'Q is an expanding relation': $E(Q) \equiv_{def} \forall x \forall y(\Phi_x Q(x, y) \supset Q(x, y))$.

(T3) Displacement: $\forall Q \forall x \forall y \forall z(Obj_x(y) \wedge E(Q) \wedge \Phi_x Q(x, y) \wedge \mathcal{E}(z) \wedge \Phi_x(u = z) \mapsto Q(x, z))$.

Analysis of the example. To return to Marvel's story, let m be Marvel, f be 'Father', l be 'Little Marvel', h be the hamster, and $C(x)$ be a distinctive property 'to have chubby cheeks'. Let $T(x, y)$ be an expanding relation 'x takes care of and controls y'. If we assume that Marvel identifies himself with 'Father' in his phantasy, the following formulas will apply in the current situation: $\mathcal{E}(m)$, $\mathcal{E}(h)$, $\Phi_m(m = f)$, $\Phi_m T(f, l)$, $E(T)$, $Obj^C_m(l)$, $\Pi_m C(h)$. Using (T1) we get that in every resulting situation $\Phi_m(l = h)$ and $\Pi_m(l \rightsquigarrow h)$ must be true. Furthermore, (T2) and (T3) justify the truth of $\Phi_m T(m, h)$ and $T(m, h)$ in any resulting situation.

Further discussion. LPI is open to the integration of more intentional operators such as those of wish, drive, and unconscious belief, more types of intentional objects such as fictional or abstract objects, and more axioms formalising unconscious mental operations.

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Interpretations with parameters, bi-interpretability, and the ε-calculus

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This project is intended to develop and philosophically examine a generalisation of the notion of relative interpretation between first-order theories, namely interpretations with parameters, and its induced standard of equivalence, bi-interpretability with parameters. In the recent philosophical literature, considerable attention has been paid to the notion of interpretation and bi-interpretability from a philosophical perspective, investigating whether interpretability can be understood as reduction and whether bi-interpretable theories can be regarded as bona-fide equivalent (Niebergall, 2000; Button & Walsh, 2018).

Roughly, an interpretation $f : S \to T$ between two first-order theories allows one to define the concepts of S in T. From a semantic point of view, it allows for the dual construction of models of S as definable inner models of T. However, the working model theorist's notion of definability is decidedly more liberal than that studied by philosophers: Model theorists typically allow for definitions *with parameters*. The rough idea here is that model theorists permit themselves the free temporary use of additional constants standing for elements of given models in their definitions. If one generalises the notion of interpretation to allow for these definitions with parameters, one obtains the notion of interpretation with parameters. Notable and historically important examples which will be discussed include:

- The Klein-Beltrami model of hyperbolic geometry (Arcozzi, 2012)
- Tarski's interpretation of group theory in true arithmetic (Tarski et al., 1953)
- The geometrisation of arithmetic (Schwabhäuser et al., 2013)

From the syntactic point of view, an interpretation with parameters $f : S \to T$ allows the formulas interpreting the primitives of S to contain additional free variables, stipulated to be drawn from a given definable parameter domain (Szczerba, 1980; Visser, 2004). Hence, axioms of S are mapped to open formulas in the language of T, containing certain additional free variables serving as parameters, such that the theory T proves the sentence resulting from stipulating that the relevant variables take values in the parameter domain. The goal of this project is to gain a better understanding of this notion of interpretation, the induced equivalence relation on first-order theories, and the semantic status of parameters. In particular, it shall be investigated whether bi-interpretability with parameters can serve as a genuine notion of equivalence.

As a first step in this direction, the semantic dual to an interpretation with parameters is investigated. An ordinary interpretation $f : S \to T$ gives rise to a function $f^* : Mod(T) \to Mod(S)$, mapping each model of T to the model of S defined in T by f. Moreover, this function between model classes can be extended to a functor between the categories of models of S and T. An interpretation with parameters $f_\pi : S \to T$ gives rise to a family of models of S in each model of T, one for each choice of parameter in the model. This idea, originally due to Hájek (1966), will be substantially elaborated on by a definition of bi-interpretability in these terms, requiring the introduction of definable families of isomorphisms between models. Moreover, it will be shown that this semantic dual can be understood in category-theoretic terms, via the notion of a profunctor on the relevant categories of models. These profunctors can be understood as categorical generalisations of relations. Theories which are bi-interpretable with parameters do not necessarily have equivalent categories of models, but thinking of interpretations with parameters as profunctors allows one to show that the categories of relation-valued presheaves on the categories of models are equivalent, yielding a novel invariant of theories.

These technical developments will allow for a technical and philosophical evaluation of bi-interpretability with parameters from both a syntacical and a semantical point of view. A notable example of such a bi-interpretation is the one between Euclidean geometry and the theory of the real numbers, essentially proven in (Schwabhäuser et al., 2013). In some respects, bi-interpretability with parameters does not have properties as good as the usual notion of bi-interpretability; for instance, automorphism groups of models and categories of models are not preserved. This is demonstrated by the bi-interpretability between the theory of real closed fields and Euclidean geometry: The real numbers are rigid, having no definable automorphisms, and the Euclidean plane has the Euclidean group of isometries as its automorphisms groups, and yet these two models (and theories) are bi-interpretable if (and only if) one allows parameters. It will be argued, however, that bi-interpretability with parameters still allows for an important use of the ordinary notion, namely a transfer of deducibility between the relevant theories. Button & Walsh (2018) argue that this kind of transfer is the philosophically most important use of interpretability, and therefore it is important that it is not lost if parameters are allowed. Moreover, bi-interpretability with parameters still preserves many important mathematical properties of theories, such as κ-categoricity, completeness, decidability, and stability. Hence, there are good reasons to accept it as a notion of equivalence for some purposes, even if it lacks some of the desiderata of bi-interpretability without parameters.

After this preliminary investigation of the induced notion of equivalence, the focus will move to a better semantic understanding of parameters. From the syntactical perspective, they seem to be closely related to free variables, while they seem semantically closer to constants. In model theory textbooks, parameters are usually considered only in the context of single models, where they can be explicated in terms of additional constants. Moreover, the existence of an interpretation with parameters $f_\pi : S \to T$ is equivalent to the existence of an ordinary interpretation $f : S \to T_\pi$, where T_π is an extension of T with a suitable constant. Nevertheless, it will be argued that one should not think of parameters as permanently added constants: Thinking of parameters as constants makes for a worse theory of interpretations and gives rise to a coarser notion of equivalence.

Instead, taking a cue from the philosophical literature on parameters in natural deduction calculi and the literature on ante-rem structuralism in the philosophy of mathematics, in particular from Shapiro's and Leitgeb's work on reference to elements of non-rigid structures, it will be argued that parameters are best understood as "well-behaved" ε-terms in Hilbert's ε-calculus (Shapiro, 2012; Leitgeb, 2021). The ε-calculus is a proof-theoretical tool developed by Hilbert et al. (1939) in the context of Hilbert's finitist programme. However, it is of broader conceptual importance in philosophy and linguistics, as it can be used to formalise indefinite descriptions and anaphoric reference (Avigad & Zach, 2024).

Syntactically, the ε-operator is a term-forming, variable-binding operator. Semantically, it is interpreted using choice functions: Applying εx to a formula $A(x)$ results in an x such that $A(x)$, if there is such an x. The main technical goal of the project is to argue that parameters can be understood as ε-terms and thus lack the semantic uniqueness associated with constants by defining a notion of interpretation in the ε-calculus and then proving that bi-interpretability with parameters agrees with bi-interpretability in the ε-calculus. This requires significant logical work, building on extant work on the syntax and semantics of the ε-calculus, but would serve to vindicate the idea that parameters can be understood as ε-terms(Leisenring, 1969; Blass & Gurevich, 2000). Moreover, this opens a new avenue for the defender of bi-interpretability with parameters: If one can defend the ε-calculus as a generally harmless tool, for instance by showing that the ε-operator is logical, one could argue that bi-interpretability with parameters is an adequate standard of equivalence on the grounds of the logical admissibility of the ε-calculus (Woods, 2014).

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