LOGICA 2021

ABSTRACTS

Table of Contents

Invited talks

Contributed talks

Invited talks

A Unifying Program for Defeasible Reasoning Forms: Adaptive Logics

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Most actual reasoning is defeasible and all knowledge ultimately results from defeasible reasoning. The latter is clearly distinct from deductive reasoning, in technical respects as well as in philosophical respects. Defeasible reasoning forms display often an external dynamics (non-monotonicity) and always an internal dynamics. The internal dynamics results from the absence of a positive test—the consequence set need not be recursively enumerable.

Crucial is the so-called standard format. The pursued thesis is that every sensible defeasible reasoning form is characterised by an adaptive logic in standard format. The logics have a selection semantics and a dynamic proof theory, which explicates the reasoning process. A strong point of the approach is that the metatheory is studied for the whole domain at once rather than for each logic separately.

At the predicative level, defeasible reasoning forms have very weak decidability properties. This required the development of dynamic proofs, of which the usual (static) proofs are a special case.

The standard format is a concept under construction. While the phrase was used at least since 2001, even now a paper modifying the notion is under review. The related research introduces an unexpected result: while every logic **L** (phrased in terms of a syntactic inference relation) is sound and complete with respect to a multiplicity of semantic systems, it also determines a unique semantics that is 'natural' in that it delineates the situations (say, sets of true and false formulas) that are possible according to **L**.

Adaptive logics offer precise and formal characterisations of methods. While their origin lies with methods for handling inconsistency, there are many results on other methods (inductive generalisation, abduction and explanation, erotetic logic, deontic logic, etc.). Most of these require *ampliative* adaptive logics (logics that extend classical logic). Unexpectedly, it was shown that adaptive logics are also interesting for defining complex mathematical theories. While the weakness of decidability properties also affects classical theories, adaptive logics engender theories that are up to Π^1_1 complex and of which the non-triviality is provable by finitary means.

The lecture will offer a survey of the program and elucidate technical as well as philosophical results. Innovative aspects will be emphasised. Attention will be paid to the comparison with other approaches to defeasible reasoning forms.

The Genesis of the Concept of a Gaggle

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Interpretations of logics in terms of propositions stimulated both formal and informal investigations of various non-classical logics. Kripke's semantics for $\mathbf{S5}$ and some other normal modal logics, and then, his semantics for intuitionistic logics ($\mathbf{J}$) are well-known. Thinking about the representation of the Lindenbaum algebra of a logic, one might wonder why $\Box$ and $\Diamond$ get modeled from the same binary relation, just as the tense operators F and G do. Further, in the semantics of $\mathbf{J}$, implication (a binary connective) is modeled from a binary relation. The Meyer–Routley semantics for relevance logics uses a ternary relation to model the entailment connective, which is binary. However, there are other complications, for instance, the actual world is supplanted by a set of logical situations.

J. Michael Dunn introduced gaggle theory (i.e., ***g**eneralized **G**alois **l**ogics*) in a series of papers (Dunn 1991, 1993, 1995, 1996, 2001). He also co-authored two books (Dunn and Hardegree 2001, Bimbó and Dunn 2008) which deal with gaggle theory. Applications of the theory yielded new semantics and new perspectives on extent semantics.

In this talk, I will overview some pre-cursors to gaggle theory — including concrete semantics by Kripke, by Meyer and Routley, and by Dunn. Then, I will piece together an archetypical gaggle from its core components together with the matching gaggle semantics. Finally, I will show that the key elements making up a gaggle may be found in Dunn's earlier work.

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Revisting Brandom's Incompatibility Semantics

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In his Locke Lectures, published as *Between Doing and Saying*, 2008, Robert Brandom introduced a new type of semantics for classical propositional logic, augmented by an S5-style modal operator. This incompatibility semantics is embedded in the context of analytic pragmatism and logical expressivism. Quite unusually, it does not involve any notion of truth, but rather puts the pragmatic and normative notion of incompatibility at its center. Brandom argued that incompatibility semantics shows that one can embrace meaning holism without sacrificing recursive reducibility of logically complex consequence claims to atomic ones. Given these far reaching claims, it is somewhat disappointing that incompatibility semantics did not receive much attention by other logicians, so far. We argue that this is at least partly due to technical as well as conceptual problems with Brandom's setup. On a more constructive note, we will discuss variations of incompatibility semantics that address some the problems. Moreover, we will explore the possibility to generalize the concept to allow for graded incompatibility. Finally, we will argue that Brandom's main aims might be better served by a game based approach to reasoning.

On Terminating Sequent Calculi

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For some proof-theoretic proofs that make use of a sequent calculus it does not suffice that the calculus in question has good structural properties, such as the admissibility of the cut rule. Sometimes it is important that the calculus is terminating as well, which roughly means that there is a well-founded order on sequents such that in any rule of the calculus the premises come before the conclusion in that order.

Whereas one of the standard calculi for classical propositional logic, G3cp, is terminating in this sense, its intuitionistic counterpart G3ip is not. The calculus G4ip was introduced by Roy Dyckhoff in 1992 as a terminating analogue of the calculus G3ip. In this talk it is shown how this result can be extended in a uniform way to a large class of intuitionistic modal logics. The method is then applied to several well-known logics, and its use in the study of intuitionistic modal logics is illustrated with some examples.

Contributed talks

Relevant Consequence Relations

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The notion of consequence is at the very core of the logical enterprise. Meta-mathematical studies of consequence relations started with the works of Alfred Tarski (1936) and have been part of a very distinguished tradition (e.g., Rasiowa, 1974; Blok & Pigozzi, 1986; Restall, 2000; Dunn & Hardegree, 2001). However, the standard mathematical rendering of the notion of consequence has many embedded features that make it unsuitable for a natural representation of some well-known non-classical logics, such as substructural logics (e.g., Schroeder-Heister & Došen, 1993; Restall, 2000; Galatos et al., 2007). The failure of the Tarskian approach to represent the so-called 'internal consequence relation' of these logics has been pointed out, e.g., by Galatos et al. (2007, p. 78).

Following an earlier work by Avron (1988; 1991; 1992) and Cintula et al. (2016; 2019), in this contribution we put forward a refined notion of consequence relation, which is more flexible than the traditional Tarskian definition. In particular, we focus on varieties of *relevant* consequence relations. Relevant logics (e.g., Anderson & Belnap, 1975; Anderson et al., 1992; Routley et al., 1982) have sought to express logical consequence 'internally' by the provability of conditional formulas, and have placed constraints on the provable conditional formulas designed to capture the desired relevance notion. As a result, much of the existing work on relevant logics concerns them understood only as sets of formulas, and the perspective concerned with ('non-internal') logical consequence has been relatively underinvestigated. We investigate a handful of generalisations of the Tarskian definition of consequence which build in various relevance requirements, comparing them and setting out some of their properties.

A consecution in a *multiple-conclusion consequence relation,* $\Gamma \vdash \Delta$, is typically taken to imply that from all the elements of Γ, at least one of the elements of Δ (or a disjunction of finitely many of them) follows. This might be called the *disjunctive reading* of a multiple-conclusion consequence relation, and constitutes the dominant approach in the literature. However, there is another, and perhaps more straightforward, possibility which we call the *conjunctive reading:* from all the elements of Γ, all the elements of Δ follow. Clearly, in logical settings with the contraction rule (where Γ and Δ are sets), this can be reduced to a series of elementary claims $\Gamma \vdash \delta$ for each $\delta \in \Delta$; this may explain why disjunctive readings are more prevalent in the literature. Nevertheless, the conjunctive reading has also been used in classical settings, e.g., in the abstract category-theoretical study of consequence by Galatos & Tsinakis (2009); and perhaps surprisingly, the conjunctive reading can be traced back to the very origins of modern mathematical logic. In fact, as early as in the first half of the 19th century, Bernard Bolzano (2004, p. 54) introduced a notion of multiple-conclusion consequence with the conjunctive reading:

> One especially noteworthy case occurs, however, if not just some, but all of the ideas that, when substituted for $i, j, \ldots$ in $A, B, C, \ldots$ make all these true, also make all of $M, N, O, \ldots$ true [...] with respect to the variable parts $i, j, \ldots$

Our starting point is the notion of multiset consequence relations proposed recently in the literature (Cintula & Paoli, 2016; Cintula et al., 2019), which utilises a multi-conclusion format with multisets of formulas on both sides and the conjunctive reading. This notion was initially proposed to circumvent a trivializing issue against multiset consequence relations raised by Ripley (2015), although to avoid another major change to the Tarskian notion of consequence relation, it was rendered with a built-in inference rule of *weakening*. Removing the rule of weakening (and thereby admitting non-monotonic reasoning, of a sort) is necessary to obtain a properly relevant theory of logical consequence.

In this talk, we will motivate the proposed framework by considering relevance constraints on consequence, as discussed in the relevant logic tradition. We will also compare non-monotonicity in substructural and relevant logics with that in defeasible reasoning. Moreover, we will discuss suitable general conditions on non-monotonic consequence relations (including several variants of the rule of cut), as well as further changes that need to be made to related Tarskian concepts (such as the definition of the deductive closure), and present some of the mathematical properties of the appropriately modified notions.

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Probability and Degrees of Truth

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Probabilities and intermediate degrees of truth are essential tools for expanding the range of logical methods beyond classical logic, and bring them closer to real-world reasoning scenarios. Besides their superficial similarity in typically relying on the real $[0, 1]$ interval, their difference is well-understood in the literature, both conceptually and formally.

Conceptually, while probabilities are typically used to quantify the uncertainty of an agent, degrees of truth can be used to model even situations of complete knowledge on the side of the agent. Formally, probability measures typically are defined on top of classical logic and presuppose it, similarly to the epistemic modalities. On the other hand, intermediate degrees of truth arise when replacing classical logic with a different logic (typically a fuzzy one), hence taking the very objects of reasoning to be different from the usual two-valued ones adopted in mathematical reasoning. In tune with this reading, probabilities, and other measures of uncertainty (and also similar ones, such as belief functions, possibility, ranking functions (Halpern, 2003)) are require to behave in non-truth functional way, in sharp contrast with degrees of truth (Dubois and Prade, 2001).

This clear-cut distinction, however, does not exhaust the connections between the two notions. In this work, we aim at clarifying and revisiting some foundational aspects of the subtle interplay between them, discussing in particular the following issues : 1) the connection between the inferential apparatus of fuzzy logics and that for probabilistic reasoning advanced in the literature, 2) the conditions under which probabilistic reasoning can be taken to be truth-functional, at least partially 3) the implications of the former idea for the justification of the notion of degrees of truth.

For the first theme, we will explore how the apparatus of fuzzy logic can be used for capturing reasoning *about* probability. Various logical approaches to probabilistic reasoning, such as those in (Hailperin, 1996; Halpern, 2003; Adams, 1998) will be surveyed, and it will be showed to what extent they can be encoded into systems of fuzzy logic. Recent work (Baldi et al., 2020) has shown in particular a translation between the probabilistic system in Halpern (2003) and suitable modal extensions of Łukasiewicz logic, one of the main axiomatic systems in the family of mathematical fuzzy logic. Since the translation goes in both direction, we also have that inferences in Łukasiewicz logics can be encoded into suitable inferences of probabilistic logics. We will in particular highlight the helpful role played in these translations by the Gentzen-style system for Łukasiewicz logic (Metcalfe et al., 2008), whose interpretation matches well-known proposals in the literature(Adams, 1998) for probabilistic consequence relations.

Secondly, we will focus on the case where probabilities have a restricted truth-functional behaviour, following recent works in this direction, such as (Coletti and Scozzafava, 2004) and (Lawry, 2014).

As an immediate example of such phenomenon, let us consider reasoning with a random variable X, taking finitely many values, e.g. $x_1 \leq x_2, \cdots \leq x_n$. Computing the probability of, say, $P(X \leq x_i \wedge X \leq x_j)$ does not involve any uncertainty, and is, as a matter of fact a truth-functional operation. This intuition can be made more precise, by capturing the cumulative distribution of X in a finite propositional language, as follows: a) denote each $X \leq x_i$ by a proposition p_i b) Pick the *classical* theory $T = \{p_1 \vee \cdots \vee p_n\} \cup \{p_i \rightarrow p_{i+1}\}_{i=1,\ldots,n-1}$. The theory encodes that fact that $x_i \leq x_{i+1}$ for each i and the x_is exhaust all the possible values for X. Considering now probability functions over models of this theory, we easily obtain that, for any formula built from p_i via the connectives $\wedge, \vee$, probability behaves truth-functionally, i.e. $P(\varphi \wedge \psi) = min(P(\varphi), P(\psi))$ and $P(\varphi \vee \psi) = max(P(\varphi), P(\psi))$. Note that truth-functionality does not hold for the whole language, as is witnessed by the fact that e.g. $P(p_i \vee \neg p_i) = 1 \neq \min(P(p_i), 1 - P(p_i))$. A similar observation is made by (Lawry, 2014), which focuses, equivalently, on probability defined over certain properly ordered sets of classical valuations.

The truth-functionality of probability in the special case above suggests a further connection between measures of uncertainty and degrees of truth. Let us interpret the highest value x_n in the example above, as the threshold for acceptance of a certain graded expression. For instance, we might consider X standing for *being tall*, and take x_n to be 1.80 m. Under this reading, we can take a probability such as $P(X \leq x_i)$ to be a normalized measure of how much the value x_i is judged to be close to x_n. In this setting, the probability will stand for how much x_i is acceptable (even though it is not fully acceptable) rather than as the measure of an uncertain event.

The discussion above leads to the following proposal: degrees of truth can be considered as measures imposed on top of fully classical models, to keep track of the distance of objects from an acceptance threshold. This view agrees with the proposal in (Cintula et al., 2017), where degrees of truth, and the logical apparatus of fuzzy logic, is seen as suitable also for graded, but non-necessarily fuzzy predicates. An example used there is the notion of *acute angle*, which is a crisp predicate, and has a related sharp threshold its correct applicability. The notion comes nevertheless in degrees: an agent would be *less wrong* in claiming that a 91 degrees angle is acute, rather than an 180 degree one.

The idea of interpreting degrees of truth in terms of distance (or closeness) to a threshold can be used as a guide to ground some of the truth-functions behind the connectives and quantifiers of the main fuzzy logics: Both well-studied ones, such as the Łukasiewicz t-norm for conjunction, and relatively unexplored ones, can have natural justifications in this setting.

While in our setting degrees of truth are not primarily tools for capturing vague predicates, they can still be accommodated. Synthesizing ideas proposed mainly in (Smith, 2008; Fine, 1975; Fermüller and Kosik, 2006), we can think of adding a further semantic layer, and admit for vague propositions a plurality of models, each with a different threshold, to be weighted probabilistically. The truth value of a vague proposition would thus be computed in this framework as the expected closeness to a threshold, when the threshold is picked from a given probability distribution. This should address some of the main concerns with the truth-functionality of vague propositions, such as those based on penumbral connections (Fine, 1975), by treating them as arising at the global level, while at the same time preserving truth-functionality locally, within each model.

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Two-Dimensional Logics of Comparative Uncertainty

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General project. This work is a part of the project introduced in (Bílková et al., 2020). We are developing a modular logical framework for reasoning based on uncertain, incomplete or inconsistent information. In this framework, an agent is constructing their belief using probabilistic incomplete and/or conflicting information aggregated from multiple sources. We formalize such probabilistic reasoning using the framework of two-layer modal logics first introduced in (Fagin et al., 1990; Hájek, 1998) and then developed by (Cintula and Noguera, 2014) and (Baldi et al., 2020). (Bílková et al., 2020) proposed two-layer modal logics to formalise such probabilistic reasoning in a potentially paraconsistent context. These logics work roughly as follows. First, the information given by the agent's sources is given on the lower layer. It is then lifted up to the upper layer by belief modalities. Finally the reasoning with the agent's belief is encoded there.

Two-dimensional treatment of uncertainty. For the purpose of our talk, we consider agents who although not being always able to give an exact level of their certainty in some proposition, can compare their certainty in one proposition to the certainty in the other. Thus, we are interested in the expansions of Gödel logic which can be treated as the logic of comparative truth (or comparative certainty).

Two-dimensionality comes from the fact that we consider information that can be incomplete and/or inconsistent, therefore we consider logics based on expansions of the product bilattice $[0,1]\odot[0,1]$ (Avron, 1996; Avron and Arieli, 1996) where the first (resp. second) coordinate encodes the positive (resp. negative) support of a statement. While $\wedge$ and $\vee$ are defined in a standard way, there are several ways to define implication. We consider two possibilities: $\rightarrow$ dualizes implication by co-implication, and $\rightharpoonup$ understands negative support of an implication as a conjunction of the positive support of the antecedent with the negative support of the consequent. Furthemore, in each of these interpretations, we consider different possible entailments. Thus we have two families of logics. The first of them which we call $\mathsf{G}^2(\rightarrow)$ connects to one of Wansing's logic of (Wansing, 2008), namely $\mathsf{I}_4\mathsf{C}_4$, and goes back to bi-intuitionistic logic (Goré, 2000; Rauszer, 1980), the second option $\mathsf{G}^2(\rightharpoonup)$ connects to Nelson's logic $\mathsf{N4}$ (Nelson, 1949).

Definition 1 (G^2 logics)**.** For all $a,b\in[0,1]$, we set $a\wedge b := \min(a,b)$, $a\vee b := \max(a,b)$ as well as

$$a \rightarrow_G b := \begin{cases} 1, & \text{if} \quad a\leq b \\ b & \text{else} \end{cases} \qquad\qquad b \prec_G a := \begin{cases} 0, & \text{if} \quad b\leq a \\ b & \text{else} \end{cases}$$

Negation and 1 are defined as $\sim_{\mathsf{G}} a := a \to_{\mathsf{G}} 0$, and $1 := \sim_{\mathsf{G}} 0$, respectively.

Now fix a countable set Prop of propositional letters and consider the following language:

$$\phi := \mathbf{0} \mid \mathbf{1} \mid p \mid \neg\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \to \phi) \mid (\phi \prec \phi) \mid (\phi \rightarrowtriangle \phi)$$

where $p \in \mathsf{Prop}$. We define $\sim\phi := \phi \to \mathbf{0}$, and $\sim_w \phi := \phi \rightarrowtriangle \mathbf{0}$.

Let $v : \mathsf{Prop} \to [0,1] \times [0,1]$, and denote v_1 and v_2 its left and right coordinates, respectively. We extend v as follows.

$$\begin{array}{rclrcl}
v(\mathbf{0}) & = & (0,1) & v(\phi_1 \wedge \phi_2) & = & (v_1(\phi_1) \wedge v_1(\phi_2), v_2(\phi_1) \vee v_2(\phi_2)) \\
v(\mathbf{1}) & = & (1,0) & v(\phi_1 \vee \phi_2) & = & (v_1(\phi_1) \vee v_1(\phi_2), v_2(\phi_1) \wedge v_2(\phi_2)) \\
v(\neg\phi) & = & (v_2(\phi), v_1(\phi)) & v(\phi_1 \to \phi_2) & = & (v_1(\phi_1) \to_{\mathsf{G}} v_1(\phi_2), v_2(\phi_2) \prec_{\mathsf{G}} v_2(\phi_1)) \\
 & & & v(\phi_1 \rightarrowtriangle \phi_2) & = & (v_1(\phi_1) \to_{\mathsf{G}} v_1(\phi_2), v_1(\phi_1) \wedge v_2(\phi_2))
\end{array}$$

Validity and entailment are defined w.r.t. sets of designated values of the form $(x,y)^{\uparrow} = \{(z,z') : z,z' \in [0,1], z \geq x, z' \leq y\}$.

Definition 2 ($(x,y)^{\uparrow}$-validity and entailment)**.** ϕ is $(x,y)^{\uparrow}$-*valid* iff $v(\phi) \in (x,y)^{\uparrow}$ for any v. Γ $(x,y)^{\uparrow}$-*entails* ψ ($\Gamma \models_{\mathsf{G}^2_{(x,y)}} \psi$) iff for any v s.t. $v(\phi) \in (x,y)^{\uparrow}$ for all $\phi \in \Gamma$, we have $v(\psi) \in (x,y)^{\uparrow}$.

Results. We establish connections between $\mathsf{G}^2_{(x,1)}(\rightarrowtriangle)$'s and $\mathsf{N4}^{\perp}$ by (Nelson, 1949) as well as between $\mathsf{G}^2_{(x,y)}(\to)$'s and $\mathsf{I}_4\mathsf{C}_4$ from (Wansing, 2008). In particular, we show that the set of valid formulas of any $\mathsf{G}^2_{(x,y)}(\to)$ coincides with that of $\mathsf{I}_4\mathsf{C}_4{}^{\perp} + (\phi \to \psi) \vee (\psi \to \phi)$ while the set of valid formulas of any $\mathsf{G}^2_{(x,1)}(\rightarrowtriangle)$ coincides with that of $\mathsf{N4}^{\perp} + (\phi \to \psi) \vee (\psi \to \phi)$. Thus, just as G is a prelinear extension of the intuitionistic logic, G^2's are prelinear extensions of its bilattice expansions.

Furthermore, we present a unified sound and complete tableau system for all G^2's introduced in (Bílková et al., 2021). We also show the expected duality between algebraic semantics on the one hand, and prelinear frames for $\mathsf{N4}^{\perp}$ and $\mathsf{I}_4\mathsf{C}_4{}^{\perp}$ on the other hand. We use this duality to prove that any $\models_{G^2_{(x,y)}(\to)}$ is compact as long as $(x,y)^{\uparrow}$ extends $(x,1)^{\uparrow}$ or $(0,y)^{\uparrow}$ or does not contain any (z,z), and that $\models_{G^2_{(x,1)}(\rightarrowtriangle)}$'s are compact as well.

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Probabilistic Reasoning Based on Incomplete and Inconsistent Information

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Motivation. We often have access to a multitude of information coming from different sources. This information is often incomplete, conflicting and uncertain. Here we aim at formalizing how a rational agent forms beliefs based on such information.

Scenarios we have in mind can be illustrated by the following toy example. An investigator needs to know if one of the suspects was present at the crime scene. She collects information from various sources (CCTV camera recordings, ATM logs, witnesses' statements etc.). The sources of evidence confirming investigator's hypothesis (the suspect was present at the place of crime) are different from, and in general independent of, those rejecting it (there is a CCTV camera closed to the crime scene vs. ATM in a supermarket in a different city). A lack of evidence supporting the hypothesis is not a reason to reject it. In the end the investigator has to aggregate the available information and form some beliefs about what likely happened.

Probabilistic reasoning based on incomplete and inconsistent information. We take *First Degree Entailment* FDE (Anderson et al., 1992) as our base logic and study the meaning of probabilities, conditional update, belief functions and their aggregation in that framework. We base our work on *non-standard probabilities* (Klein et al., 2021) to account for potentially contradictory information about events. A *probabilistic model* is a tuple $\mathscr{M} = \langle \Sigma, \mu, v^+, v^- \rangle$ where Σ is a finite set of states, $v^+, v^- : \Sigma \times \mathrm{Prop} \to \{0,1\}$ are valuations representing respectively the positive and negative information and μ is a probability measure on the powerset algebra $\mathscr{P}(\Sigma)$. Let $|\varphi|^+_{\mathscr{M}} = \{s \in \Sigma : v^+(\varphi) = 1\}$ and $|\varphi|^-_{\mathscr{M}} = \{s \in \Sigma : v^-(\varphi) = 1\}$. The *non-standard probability function* based on $\mathscr{M}$ is the couple of maps (p^+_μ, p^-_μ) where $p^+_\mu(\varphi) := \mu(|\varphi|^+_{\mathscr{M}})$ (resp. $p^-_\mu(\varphi) = \mu(|\varphi|^-_{\mathscr{M}})$) represents the positive (resp. negative) probabilistic evidence for φ. Non-standard probabilities satisfy the following axioms: (i) if $A \vdash_{\mathsf{FDE}} B$ then $p^+(A) \leq p^+(B)$, (ii) $p^+(A \wedge B) + p^+(A \vee B) = p^+(A) + p^+(B)$, and (iii) $p^+(\neg A) = p^-(A)$. Notice that one can no longer prove that $p^+(\varphi) + p^+(\neg\varphi) = 1$. Indeed the two values are independent here.

Belief functions interpreted over FDE. To handle cases where available information does not allow to define the probability of some formulas (i.e. some subsets of Σ are non-measurable), we define *partial probabilistic models* $\mathscr{M} = \langle \Sigma, \mathscr{X}, \mu, v^+, v^- \rangle$, where $\mathscr{X}$ is the σ-algebra of measurable subsets of Σ. Notice that p_μ^+ and p_μ^- are partial functions on the measurable formulas. We use inner and outer measures to reason about non-measurable elements as follows

$$(p_\mu^+)_*(\varphi) = \sup\{p_\mu^+(\psi) : |\psi|^+ \subseteq |\varphi|^+ \text{ and } |\psi|^+ \in \mathscr{X}\}.$$

Another way of reasoning with non-standard probabilities covering the non-measurable elements is to use belief functions instead of probability measures. The above method is a special case of the latter case, based on the fact that inner (resp. outer) measures are belief (resp. plausibility) functions.

Conditional update for non-standard probabilities. (Klein et al., 2021) already propose several kinds of conditioning over non-standard probabilities. We propose two new definitions in order to be able to talk about conditioning on partial probabilistic models.

The classical standard conditioning is not able to talk about non-measurable elements. We generalise the standard conditioning to non-Boolean structures containing non-measurable elements as follows. The positive and negative probability structures associated to a partial probabilistic model are $\langle \mathscr{L}, L, \eta^+ \rangle$ and $\langle \mathscr{L}^{op}, L^{op}, \eta^- \rangle$ where $\mathscr{L}$ is the Lindenbaum algebra, L and L^{op} are respectively the sublattices $\{[\varphi] : |\varphi|^+_{\mathscr{M}} \in \mathscr{X}\}$, $\{[\varphi] : |\varphi|^-_{\mathscr{M}} \in \mathscr{X}\}$ and $\eta^+([\varphi]) = p_m u^+(\varphi)$ and $\eta^-([\varphi]) = p_\mu^-(\varphi)$. Let $M = \langle \mathscr{L}, L, \eta \rangle$ be a probability structure, $a \in L$ and $\eta(a) \neq 0$. Conditioning on a gives rise to the probability structure $M_a = \langle \mathscr{L}_a, L_a, \eta_a \rangle$ where $\mathscr{L}_a$ is the congruence lattice based on a, $L_a = \{[c] : c \in \mathscr{L} \text{ and } c \wedge a \in L\}$ is a sublattice of $\mathscr{L}_a$, and η_a is defined over L_a as follows: $\eta_a([c]) = \frac{\eta(c \wedge a)}{\eta(a)}$. For every $c \in \mathscr{L}$:

$$(\eta_a)_*([c]) = \frac{\eta_*(a \wedge c)}{\eta(a)} \quad \text{and} \quad (\eta_a)^*([c]) = \frac{\eta^*(a \wedge c)}{\eta(a)}.$$

We also propose a second way of conditioning following (Fagin and Halpern, 1991) which is based on the fact that belief functions are the lower envelopes of the the probability measures consistent with the given belief function.

Future directions and context. We develop the duality between probabilistic models and non-standard probabilities over de Morgan algebras and to adapt standard definitions and tools of (imprecise) probability theory to non-classical reasoning. In addition we still need to fully understand the philosophical meaning of non-standard probabilities, their associated belief functions and conditional updating.

This work is part of a wider project aiming at developing a modular logical framework to formalize probabilistic reasoning based on incomplete and/or inconsistent information (Bílková et al., 2020). We propose a *two-layer modal logical framework* (Baldi et al., 2020). The bottom layer is to be that of events or facts, represented by probabilistic information provided by sources available to an agent with a certain degree of reliability. The modalities connecting bottom layer to the top layer, are that of belief of the agent (e.g. about an event taking place) based on the information from the sources in terms of (various kinds of) aggregation. The top layer is to be the logic of thus formed beliefs. This work focuses on the semantics of the modalities that connect the bottom and the upper layer.

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Simple Semantics for Logics of Indeterminate Epistemic Closure

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Jago (2014) has developed a novel solution to the problem of logical omniscience, but his epistemic logic is not axiomatized, the semantics is difficult to navigate, and it is motivated in part by substantive metaphysics. In this paper, I show how to capture his results more simply by adapting the hyperintensional modal semantics of Sedlár (forthcoming).

According to Jago (2014, pp.206-207), some logical consequences are 'trivial' in the sense that any agent recognizes them—for example, the primitive rules of classical propositional logic (CPL). We cannot deny that knowledge is closed under trivial consequences. The problem is that this is like a *tolerance principle* for knowledge closure, pushing us toward the conclusion that knowledge is fully closed under all of its logical consequences (omniscience). So, we have a paradox. The norms of knowledge ascription push in two conflicting directions: we are compelled to accept trivial closure while denying full closure. Jago's epistemic logic addresses this paradox by appeal to *indeterminate barriers* to knowledge closure. His logic has a determinacy operator Δ that interacts with knowledge in the following way.

- **Indeterminate Barrier:** if $\varphi \vDash \psi$ is trivial, then $\Delta K\varphi$ implies $\neg\Delta\neg K\psi$

This allows us to say that there are barriers to knowledge closure without denying the tolerance principle for trivial closure—because it is never determinate that an agent has knowledge *and* that the agent fails to know something that follows trivially. My ultimate goal in this paper is to replicate Jago's epistemic logic in an extension of Sedlár's framework. I will only sketch the core result for now: the Barrier Theorem.

A single-agent Sedlár model $M = \langle W, C, H, I, N\rangle$ contains a set of possible worlds $W \neq \emptyset$ and a set of fine-grained contents $C \neq \emptyset$. We think of the elements $c \in C$ as abstract entities that individuate the cognitive roles of sentences, without further elaboration. The *hyperintension* function $H : \text{Form} \to C$ assigns a fine-grained content to each formula, whereas the *intension* function $I : C \to \wp(W)$ assigns a coarse-grained proposition (set of worlds) to each content. This gives rise to hyperintensionality because knowledge is defined over fine-grained contents of sentences rather than their coarse-grained propositions. There is a neighborhood function $N : W \to \wp(C)$ used to interpret knowledge. It assigns a set of known contents $N(w)$ to the agent at w. For the factivity of knowledge, we also assume that neighborhoods and intensions are related as follows: if $c \in N(w)$, then $w \in I(c)$.

The proposition $[\![\varphi]\!]^M \subseteq W$ expressed by φ in M is defined:

- $[\![p]\!]^M = I(H(p))$ for atomic p
- $[\![\bot]\!]^M = \emptyset$
- $[\![\neg\varphi]\!]^M = W \setminus [\![\varphi]\!]^M$
- $[\![\varphi \to \psi]\!]^M = (W \setminus [\![\varphi]\!]^M) \cup [\![\psi]\!]^M$
- $[\![K\varphi]\!]^M = \{x \in W \,:\, H(\varphi) \in N(x)\}$

I extend this framework with one additional ingredient that is needed for Jago's determinacy operator. A sphere model $M = \langle W, C, H, I, N, S \rangle$ is a Sedlár model extended by an additional function $S : W \to \wp(W)$ that assigns a 'sphere of alternatives' $S(w)$ to each world w. These are ways of *sharpening* the truth. The relation $x \in S(w)$ is an equivalence relation. Determinate truth is understood as what is true throughout the sphere of alternative sharpenings.

- $[\![\Delta\varphi]\!]^M = \{x \in W \, : \, S(x) \subseteq [\![\varphi]\!]^M\}$

In sphere models, determinate truths about knowledge depend on the structure of all of the neighborhoods of alternative sharpenings. For any $\Gamma \cup \{\varphi\} \subseteq$ Form, call the following condition the Γ-φ *barrier schema* (schematic over w).

$$(\boldsymbol{\Gamma\text{-}\varphi\text{-}BS}) \quad H(\Gamma) \subseteq \bigcap\{N(x) \, : \, x \in S(w)\} \wedge H(\varphi) \notin \bigcup\{N(x) \, : \, x \in S(w)\}$$

If this condition holds at w, then at w it is determinate that the agent knows all of the Γs and it is determinate that the agent does not know φ. This is *not allowed for trivial consequences*. In positive terms, we say that a model 'respects' a given consequence $\Gamma \vDash_{\text{CPL}} \varphi$ if the Γ-φ barrier schema fails at all of its worlds. Hence, we can define a class of sphere models $\mathbb{C}$ that respects any chosen set of trivial consequences as follows:

$$\mathbb{C} = \{M \, : \, \text{if } \Gamma \vDash_{\text{CPL}} \varphi \text{ is trivial } \Rightarrow (\Gamma\text{-}\varphi\text{-BS}) \text{ fails at all } w \in W\}$$

Assuming that trivial consequences are a proper subset of CPL consequences, we can validate the Indeterminate Barrier principle for any set of trivial consequences.

Theorem 1 (Barrier Theorem)**.** If $\Gamma \vDash_{\text{CPL}} \varphi$ is trivial, $\{\Delta K\psi \, : \, \psi \in \Gamma\} \vDash_{\mathbb{C}} \neg\Delta\neg K\varphi$.

This theorem is, however, quite general. It tells us that a given class of models validates a version of the Indeterminate Barrier principle. The philosophical question is: which inferences are really trivial? In other words, how can we use this to produce a correct epistemic logic? The rest of the paper will elaborate on this question and other details of Jago's theory, which actually results in a spectrum of epistemic logics (not a single unique logic). Finally, I will consider whether the 'tolerance principle' mentioned at the beginning is really important.

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Tractable Depth-Bounded Approximations to First-Degree Entailment

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Many interesting propositional logics are likely to be computationally intractable, so we cannot expect a real agent (human or artificial) to be always able to recognize in practice that a certain conclusion follows from a given set of assumptions. This is a source of difficulties in areas—such as AI, philosophy, cognitive science and economics—that need less idealized, yet theoretically principled, models of rationality. The "depth-bounded" approach to Classical Propositional Logic (e.g., D'Agostino & Floridi, 2009; D'Agostino, Finger & Gabbay, 2013; D'Agostino, 2015) provides an account of how this logic can be approximated in practice by less idealized, resource-bounded agents via its reinterpretation in terms of an intuitive, albeit non-deterministic, 3-valued semantics that was first put forward by W.V.O. Quine and whose values have a natural informational interpretation. This approach leads to an infinite hierarchy of tractable depth-bounded approximations, which can be naturally related to the inferential power of the agents and admit of an elegant proof-theoretical characterization.

The logic of *First-Degree Entailment* (**FDE**) (Anderson & Belnap, 1962) also admits of an intuitive semantics based on informational values (Dunn, 1976; Belnap, 1977), which was put forward as the logic in which "a computer should think". These values are interpreted as four possible ways in which an atom p can belong to the present state of information of a computer's database, which in turn is fed by a set of equally "reliable" sources: **t** means that the computer is told that p is true by some source, without being told that p is false by any source; **f** means the computer is told that p is false but never told that p is true; **b** means that the computer is told that p is true by some source and that p is false by some other source (or the same in different instants); **n** means that the computer is told nothing about the value of p. In turn, the values of complex formulae are computed using 4-valued (deterministic) truth-tables which are derived by monotonicity considerations.

Despite its informational flavour, **FDE** is co-NP complete (see Urquhart, 1990) and thus a highly idealized model of how an agent *can* think. The key observation in this paper is that a fair amount of idealization is still present in the standard interpretation of the values **t**, **f** and **n**, that presupposes complete information about the set of sources S by an agent a. While the meaning of **b** is "*there is* at least a source assenting to p and at least a source dissenting from p"—which is information empirically accessible to a in the sense that a may actually hold this information without a complete knowledge of S—the meaning of **t**, **f** and **n** involves information of the kind "*there is no* source such that..."—and so requires complete information about the sources in S, which may not be empirically accessible to a. What if the agent has no such complete knowledge about the sources (e.g., is receiving information from an "open" set of sources)?

Inspired by D'Agostino (1990) and Fitting (1991, 1994), we address this issue by shifting to *signed* formulae, where the signs express *imprecise* values associated with two distinct bipartitions of the standard set of 4 values. These are expressions of the form $\mathsf{T}A$, $\mathsf{F}A$, T^*A

and F^*A, where A is an (unsigned) formula. Their intended meaning respectively is: "A is at least true" (expressing that the value of A is in $\{\mathbf{t}, \mathbf{b}\}$); "$A$ is non-true" (saying that the value of A is in $\{\mathbf{f}, \mathbf{n}\}$); "$A$ is non-false" (the value of A is in $\{\mathbf{t}, \mathbf{n}\}$); and "$A$ is at least false" (the value of A is in $\{\mathbf{f}, \mathbf{b}\}$). Note that, signed formulae of the form $\mathsf{T}A$ or F^*A express information that an agent may actually hold even without a complete knowledge of the set sources S—but this is not the case of the other two types of signed formulae that involve complete knowledge of S.

We then define a hierarchy of tractable approximations to **FDE** based on such imprecise values. First, we define *linear* introduction and elimination (intelim) rules with signed formulae as premises and as conclusions. These rules generate sequences of signed formulae and we show that the consequence relation characterized by such intelim sequences is *tractable*. Next we define two *branching* structural rules according to which we are allowed to (i) append both $\mathsf{T}A$ and $\mathsf{F}A$ as sibling nodes to the last element of any intelim sequence; (ii) append both T^*A and F^*A in a similar way. The intuitive meaning of these rules is that one of the two cases must obtain even if the agent has no actual information about which is the case. In this sense, we call the information expressed by each of the two complementary signed formulae "virtual information"; i.e., hypothetical information that the agent does not actually hold. The more virtual information needs to be invoked via these branching rules, which we call *PB* and *PB**, the harder the inference is for the agent. Thereby, the nested applications of *PB* and *PB** provide a sensible measure of inferential *depth*. This naturally leads to defining an infinite hierarchy of tractable depth-bounded approximations to **FDE**, in terms of the maximum number of nested applications of *PB* and *PB** that are allowed. Note that, (i) unlike the branching rules of standard tableau methods, our branching rules are *structural* in that do not involve any specific logical operator; (ii) the elimination rules, together with the branching rules, were early introduced in (D'Agostino, 1990) as constituting a refutation method for *full* **FDE** called RE_{fde}. However, our intelim method can be used as a direct proof method as well as a refutation method and leads to more powerful approximations.

Last but not least, the informational interpretation of our intelim method can be formalized in terms of a 5-valued *non-deterministic* semantics (see Avron & Zamansky, 2011). This semantics essentially takes the signs as imprecise values, i.e. two-element sets of the standard values. Besides, a fifth value, $\perp$, is taken to represent the case where the value of a formula is completely undefined in that the agent??s information is insufficient to even establish any of the imprecise values. We finally show that all the defined approximations are tractable.

This research was partially funded by the Israel Science Foundation (grant 550/19).

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An Abelardian Reply to the Stone Problem

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Reflection on the concept of omnipotence raises puzzling questions which concern whether omnipotence is a coherent notion. The Stone problem exemplifies the kind of puzzling questions surrounding that notion:
(1) Either God can create a stone which God cannot lift or God cannot create such a stone.
(2) If God can create a stone which God cannot lift, then God cannot do everything.
(3) If God cannot create a stone which God cannot lift, then God cannot do everything.
(4) If God cannot do everything, God is not omnipotent
(5) Therefore, God is not omnipotent.

Roughly, there are two classes of responses to this problem. The first ones are what I call *theory-driven* responses (cf. (Sobel, 2004), Hoffman and Rosenkrantz (2010), Wainwright (2010)). They consist in refining core concepts whether in the notion of omnipotence —such as action, possibility, capacity— or related to the notion of God —such as limitation, perfection—. The second one are *logic-driven* responses. According to the latter, some of the principles of classical logic invoked in the argument are not valid; hence, the conclusion may be untrue, and then the argument is either invalid or unsound. See for example Beall and Cotnoir (2017), Tedder and Badía (2018).

In Beall and Cotnoir's (henceforth, B&C) (Beall and Cotnoir, 2017) logic-driven response, premise (1) is rejected, that is, the logical validity of $A \vee \sim A$ is rejected. In particular, they assume that the logic $\mathbf{K}_3$ is the right one to reason about issues like these. In their picture, each premise is neither true nor false; they are *gappy*, for short. This is so because B&C are working on top of $\mathbf{K}_3$: that (1) is gappy means that both disjuncts are gappy as well. Then, if (5) is false, (2), (3) and (4) are all gappy. All this can be easily checked with the truth tables for $\mathbf{K}_3$:

A	$\sim A$
$\{1\}$	$\{0\}$
$\{\ \}$	$\{\ \}$
$\{0\}$	$\{1\}$

$A \vee B$	$\{1\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
$\{\ \}$	$\{1\}$	$\{\ \}$	$\{\ \}$
$\{0\}$	$\{1\}$	$\{\ \}$	$\{0\}$

(Recall that '$\{1\}$' stands for *is assigned truth only*, '$\{0\}$' stands for *is assigned falsity only* and '$\{\ \}$' stands for *is assigned neither truth nor falsity*, whereas $A \rightarrow B$ is defined as $\sim A \vee B$)

In this paper, I want to explore another logic-driven response to the Stone Problem, one in which every premise but (2) is true. Said briefly, the motivation for that option is as follows. Whatever the values of the components, $A \vee \sim A$ should be a logical truth, so (1) must be true. (3) is true because a good many conditionals with untrue antecedents are true, and (3) meets the conditions to be so. For that very same reason, (4) is true as well. Nonetheless, (2) does not fulfill the conditions to be a true conditional even if its antecedent is untrue. What I have in mind are certain ideas by Peter Abelard regarding the negative determination of natures, as in "God (with the property $P(x)$) *is not* $Q(x)$", which are treated as untrue conditionals by Abelard. See the reconstruction of his ideas in (Martin, 2004).

The plan of the paper is as follows. In Section 1, I present the Stone problem, the two broad kinds of responses that can be offered and B&C's logic-driven proposal. In Section 2, I explore two extensions of B&C's proposal so that (1) is untrue while both (2) and (3) are true, according to the idea that conditionals with untrue antecedents must be true. I show that, as anticipated in (Tedder and Badía, 2018), all they fall prey to Curry-like paradoxes, which would trivialize the theory (i.e. every sentence in the theory would be provable). In Section 3, I motivate and present in detail another option in the logical space, namely, that (1), (3) and (4) are true, but (2) is untrue. Roughly, the connectives would need to be evaluated as follows:

A	$\sim A$
$\{1\}$	$\{0\}$
$\{\,\}$	$\{\,\}$
$\{0\}$	$\{1\}$

$A \vee B$	$\{1\}$	$\{\,\}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
$\{\,\}$	$\{1\}$	$\{1\}$	$\{\,\}$
$\{0\}$	$\{1\}$	$\{\,\}$	$\{0\}$

$A \to B$	$\{1\}$	$\{\,\}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{\,\}$	$\{0\}$
$\{\,\}$	$\{1\}$	$\{\,\}$	$\{1\}$
$\{0\}$	$\{1\}$	$\{1\}$	$\{1\}$

$A \to_a \sim B$	$\{1\}$	$\{\,\}$	$\{0\}$
$\{1\}$	$\{\,\}$	$\{\,\}$	$\{0\}$
$\{\,\}$	$\{\,\}$	$\{\,\}$	$\{0\}$
$\{0\}$	$\{\,\}$	$\{\,\}$	$\{\,\}$

where the evaluations for $A \to_a \sim B$ are used exclusively for conditionals with positive antecedent —i.e. a formula where the main connective is not $\sim$— and negative consequent —i.e. a formula where the main connective is $\sim$ and B is not negative again—, whereas any other conditional is evaluated as in $A \to B$.

Finally, in Section 4 I discuss some possible objections and further consequences of the proposal, for example, regarding the shape of the logic underlying the true theology. Finally, an appendix is offered for the true aficionado to the technical details of the logical apparatuses discussed.

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Formal Explanation, Classical Logic, and Intuitionistic Logic

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The act of proving a sentence is usually associated with the question whether the sentence is true or false. The existence of a proof guarantees that the sentence is true, but some proofs are more informative than others. It happens sometimes that a proof stands out among the other proofs of a sentence because it does not only certify the truth of the sentence, but it also displays in the clearest way the reasons of its truth. In other words, such a proof *explains why* the sentence is true. The idea that certain proofs can be considered as rigorous explanations of the reason why a truth holds goes far back in the history of philosophy, and investigations on this notion of explanation has its origins in Aristotle's *Posterior Analytics* (Barnes, 1984, Post. An. I, 2–8) and has been carried on by Bolzano (2014) in his *Theory of Science*.

In the contemporary literature, an explanatory relation is receiving considerable attention in several fields of philosophy: the grounding relation. Grounding—which can be traced back to Bolzano's notion of *Abfolge* (Bolzano, 2014, §162, §168, §198–221)—is usually introduced as an objective and explanatory relation that connects two relata—the ground and the consequence—if the first constitutes a reason why the second holds, see, e.g., Betti (2010); Schnieder (2011); Fine (2012). In other terms, we can say that the consequence holds *in virtue of* the ground. Much work is being devoted to characterise the relation that holds between a logically complex formula F and the formulae in virtue of which F holds, see, e.g., Schnieder (2011); Fine (2012); Correia (2014); Poggiolesi (2016). Different proof systems have been developed to formally define logical grounding, see, e.g., Schnieder (2011); Correia (2014); Prawitz (1974); and while the methods of proof-theory are not yet considered standard tools for the investigation of grounding, promising endeavours of proof-theoretical analysis of grounding made their appearance in Rumberg (2013); Prawitz (1974, 2019). Nevertheless, no in-depth study focusing on the nature and characteristics of grounding rules, and on their relationship with logical rules, exists. Some even contest the legitimacy of considering logical grounding as different from logical consequence, see McSweeney (2020).

In this work, we will show that there is a sensible notion of grounding which is based on Bolzano's *Abfolge* and is clearly, formally distinct from logical consequence. We will show that this grounding relation can indeed be considered a derivability relation of a particular kind—as Bolzano argued—and we will present a thorough study of this relation by proof-theoretical means. We will focus, in particular, on the differences and interplay between grounding rules and logical rules, and we will both consider classical logic and intuitionistic logic.

We present, first, the calculus Gr, which is, at the same time, a grounding calculus and a calculus for classical logic. By using Gr, we can construct grounding derivations, logical derivations and derivations combining logical and grounding steps. Moreover, for any derivation constructed in this calculus, we can determine which of its parts are explanatory and which are purely logical. In order to study the direct interplay between grounding rules and logical rules we then conduct a proof-theoretical analysis by defining a normalisation procedure for Gr derivations, by proving that it is terminating, and by showing that it yields normal derivations enjoying the subformula property. The normalisation result guarantees that grounding rules can be suitably seen as introduction rules, in the sense that the information required to apply grounding rules is sufficient to define the meaning of the introduced connective. The subformula property guarantees in turn that the normalisation procedure is correctly defined.

This result leaves a question open, though. The calculus Gr, indeed, does not only contain grounding rules for introducing connectives, it also contains one logical introduction rule. The grounding rules for negation, indeed, are too strict to be satisfactory logical introduction rules. In Gr we adopt the simple solution to also include an unrestricted version of the negation introduction rule, and we thus obtain a complete and normalising calculus that enables us to conduct an exhaustive proof-theoretical study of the interplay between grounding and logical rules. It is still not immediately apparent whether a conceptually subtler solution is available. Could we not find a more meaningful way to reintegrate what grounding rules lack with respect to logical introduction rules? Are there logical rules, possibly elimination rules, that give us some intuition about what grounding rules are precisely missing in order to become a sufficient set of logical introduction rules? We positively answer this question by defining a calculus for grounding that only contains grounding rules and logical elimination rules. The calculus essentially relies on the presence of a rule for the excluded middle law $A \vee \neg A$. This answer to the question about what grounding rules miss with respect to logical introduction rules is essentially tied to the nature of grounding as an explanatory relation. Indeed, the formal role of the excluded middle rule in this calculus can be given a precise philosophical meaning which is tightly connected with the nature of grounding rules.

A further question arises from this result. Since classical logic can be defined by adding the law of excluded middle to intuitionistic logic, and since the addition of this law to a set of grounding rules and elimination rules yields a calculus for classical logic, do grounding rules constitute a complete set of introduction rules for intuitionistic logic? A positive answer to this question would agree with a quite widespread belief that grounding should correspond to some kind of constructive reasoning. Indeed, Bolzano himself (Bolzano, 2014, §530) stressed that explanations should not feature arguments by *reduction ad absurdum*—which is intuitionistically equivalent to the excluded middle law. Unfortunately, this is not the case. While it is true that grounding derivations do not contain non-constructive reasoning steps; as we will show, grounding rules as they are do not constitute a complete set of introduction rules for intuitionistic logic either—that is to say, if we extend our set of grounding rules by the corresponding elimination rules for intuitionistic logic, we do not obtain a complete calculus for the logic. The additional intuitionistic principles required to regain all strength of intuitionistic logic point at a correspondence between grounding and derivability in substructural logics.

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Forms of Rules Imply Disjunction Property

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It is a well known fact that the intuitionistic logic enjoys the disjunction property (DP), i.e., if $A \vee B$ is provable then either A is provable or B is provable. In this talk we are interested in the disjunction property of intuitionistic modal logics. They are obtained by adding modalities to the intuitionistic logic. Since $\Box$ and $\Diamond$ are not dual of each other in an intuitionistic setting, there are several ways of defining intuitionistic modal logics, see e.g., (Amati-Pirri, 1994), (Servi, 1980) and (Ono, 1977).

In this talk, we present a uniform method to prove DP for various intuitionistic modal logics. More specifically, we prove that if the rules in a sequent calculus for an intuitionistic modal logic have a special form, then the sequent calculus enjoys DP. As a consequence, we uniformly prove that the sequent calculi for intuitionistic logic, the intuitionistic version of several modal logics including K, T, K4, S4, S5, their Fisher-Servi versions, propositional lax logic, and many others have DP. Our method also provides a way to prove negative results: any intermediate modal logic without DP does not have a calculus of the given form. These negative results are in line with what we call *universal proof theory* project where the relation between general forms of sequent calculi and the logical properties (such as Craig and uniform interpolation) are investigated, see (Iemhoff, 2019) and (Akbar Tabatabai-Jalali, 2018).

In fact, our result is even stronger and we can prove the *feasible DP* for the mentioned logics. A proof system (e.g., a Gentzen- or Hilbert-style system) $\mathbf{P}$ for a logic is said to have feasible DP if there is a poly-time algorithm that given a proof π of the formula $A \vee B$ in $\mathbf{P}$, it outputs either a proof of A or a proof of B in $\mathbf{P}$. (Buss-Mints, 1999) and later (Buss-Pudlak, 2001) addressed the computational complexity of DP in intuitionistic logic and proved that it is feasible. Later, (Ferrari et al, 2002) and (Ferrari et al, 2005) studied the same problem and provided a framework to prove the feasibility of DP for proof systems for intuitionistic logic and some intermediate and intuitionistic modal logics. Another type of feasible DP suitable for the classical modal logics has been investigated in (Bílková, 2006) and for Frege systems for any extensible modal logic in (Jeřábek, 2006).

In the following, we will state the result more precisely. We work with the language $\mathcal{L} = \{\wedge, \vee, \rightarrow, \top, \bot, \Box, \Diamond\}$. The sequent calculus $i\mathbf{K}$ is defined by the usual sequent calculus for intuitionistic logic, $\mathbf{LJ}$, plus the rules

$$\frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\ (K_\Box) \qquad \frac{\Gamma, A \Rightarrow B}{\Box\Gamma, \Diamond A \Rightarrow \Diamond B}\ (K_\Diamond)$$

Definition.

- The set of *basic* formulas is the smallest set containing atomic formulas, the constants $\top$ and $\bot$ and closed under $\{\wedge, \vee, \Diamond\}$.

- The set of *almost positive (a.p.)* formulas is the smallest set containing basic formulas and closed under $\{\wedge, \vee, \Box, \Diamond\}$ and implications of the form $A \to B$, where A is basic and B is almost positive.
- The set of *almost negative (a.n.)* formulas is the smallest set containing basic formulas and closed under $\{\wedge, \Box\}$ and implications of the form $A \to B$, where A is almost positive and B is almost negative.

Almost positive rules. A rule is called *left almost positive*, when it is of the form

$$\frac{\{\Gamma, \overline{N'_i} \Rightarrow \overline{M'_i}, \Delta\}_{i=1}^{n}}{\Gamma, \overline{M} \Rightarrow \Delta}$$

where $\overline{M}, \overline{M'_i}, \overline{N}, \overline{N'_i}$ are multisets of formulas, $\overline{M}$ and $\overline{M'_i}$ only consist of a.p. formulas and $\overline{N}$ and $\overline{N'_i}$ only consist of a.n. formulas, for $1 \leq i \leq n$. Moreover, if $n > 1$, then all formulas in $\overline{N'_i}$ are basic (i.e., only when $n = 1$, the formulas in $\overline{N'_1}$ can be a.p. formulas that are not basic). A *right almost positive* is defined similarly.

Definition. (*Harrop formulas*) The set of *Harrop* formulas is the smallest set including atomic formulas, $\bot$, $\top$, and is closed under $\{\wedge, \Box\}$ and under implications of the form $A \to B$ where A is an arbitrary formula and B is Harrop (specially $\neg A$ is Harrop for any A).

Definition. (*Feasible Visser-Harrop property*) We say a sequent calculus $\mathbf{G}$ has *feasible Visser-Harrop property*, if there exists a poly-time algorithm that reads a proof π of $\Gamma, \{A_i \to B_i\}_{i \in I} \Rightarrow C \vee D$ in $\mathbf{G}$, where Γ is a multiset consisting of Harrop formulas and outputs a $\mathbf{G}$-proof either for $\Gamma \Rightarrow C$ or $\Gamma \Rightarrow D$ or $\Gamma \Rightarrow A_i$, for some $i \in I$.

Definition. A calculus $\mathbf{G}$ is called *T-free* if it is valid in the irreflexive Kripke frame of one node. It is called *T-full* if it is valid in the reflexive Kripke frame of one node and extends $i\mathbf{K} + \{T_a, T_b\}$, where

$$\frac{\Gamma \Rightarrow \Box A, \Delta}{\Gamma \Rightarrow A, \Delta}\ (T_a) \qquad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \Diamond A, \Delta}\ (T_b)$$

Main Theorem. Let $\mathbf{G}$ be a T-free or a T-full calculus extending $i\mathbf{K}$ and consisting only of almost positive rules, the cut rule and Nec. Then, $\mathbf{G}$ has feasible Visser-Harrop property.

Positive applications. The sequent calculi $i\mathbf{K}$, $i\mathbf{KT}$, $i\mathbf{KB}$, $i\mathbf{K4}$, $i\mathbf{K5}$, $i\mathbf{KBT}$, $i\mathbf{S4}$, $i\mathbf{KB4}$, $i\mathbf{K45}$, $i\mathbf{S5}$, their Fisher Servei versions and their $\Diamond$-free counterparts have feasible Visser-Harrop property and hence feasible DP.

Negative applications. Let $i\mathbf{K} \subseteq \mathsf{L}$ be a logic without disjunction property (e.g., any consistent modal extension of $\mathsf{LC} + i\mathbf{K}$). Then, L has no calculus consisting of almost positive rules, the cut rule and Nec.

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Definition and Computation

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A standard way of introducing new terms in science is by means of nominal definition: a new term is introduced and stipulated to be identical to a composition of already introduced terms. Following a now standard terminology, we shall call the lefthand side of a definitional equation 'definiendum' and the righthand side 'definiens'. When operating with a term introduced as a definiendum, one typically needs to go back to the corresponding definiens. For instance, to establish that a structure $\mathfrak{G}$ forms a group, one needs to establish that $\mathfrak{G}$ satisfies all of the usual group conditions. If the definiens involves a term that is itself nominally defined, one might have to look in turn at its definiens. For instance, to establish that the triple $\mathfrak{G}$ is a Lie group, one needs to establish, in particular, that $\mathfrak{G}$ is a group, and this requires that one looks at the definition of 'x is a group'.

This process of unfolding definitions, as we might call it, has a long and distinguished history in logic. Aristotle appeals to it in a number of places in his *Topics*, and it played a fundamental role in Leibniz's logic. In the first part of my talk, I wish to argue that this process has a role to play also outside the theory of definition narrowly construed. Namely, I wish to argue that computation, or calculation, can naturally be conceived of as the unfolding of definitions. A connection between computation and definition is in fact hinted at already in Leibniz's writings, since he calls the definiens of a definiendum its value ('valor' in Latin). In the second part of the talk I will consider how this notion of computation fits with the notion of computation as conversion in combinatory logic and lambda calculus.

In mathematical logic one usually applies the adjectives 'computable' and 'calculable' to functions. It is, however, not the function in isolation that is operated on in any given computation, but rather the function completed by a suitable sequence of arguments. For instance, we do not compute the addition function in isolation, but we may compute $2+2$, the result of completing the addition function with two arguments from its domain of definition.

Let me illustrate how the computation of $2+2$ proceeds when computation is understood as the unfolding of definitions. In formalizations of arithmetic, the term '2', if it occurs at all, will usually be defined as $\mathbf{s}(1)$, and the term '1' will be defined as $\mathbf{s}(0)$, where $\mathbf{s}$ is the successor function. The terms 's' and '0' are not nominally defined, but are primitive (how they are to be explained will not be discussed in this talk). The term '$2+2$' moreover contains the symbol '+', which we may also regard as nominally defined, namely by the pair of equations,

$$m + 0 \equiv 0$$
$$m + \mathbf{s}(n) \equiv \mathbf{s}(m+n)$$

The definition made up by these two equations is not an explicit, or purely abbreviatory, definition, but it is natural nevertheless to class it together with such definitions as another species of nominal definition. By unfolding these definitions, we may compute $2+2$ as follows:

$$2+2 \underset{\text{Def of 2}}{\longrightarrow} 2+\mathbf{s}(1) \underset{\text{Def of +}}{\longrightarrow} \mathbf{s}(2+1) \underset{\text{Def of 1}}{\longrightarrow} \mathbf{s}(2+\mathbf{s}(0)) \underset{\text{Def of +}}{\longrightarrow} \mathbf{s}^2(2+0)$$
$$\underset{\text{Def of +}}{\longrightarrow} \mathbf{s}^2(2) \underset{\text{Def of 2}}{\longrightarrow} \mathbf{s}^3(1) \underset{\text{Def of 1}}{\longrightarrow} \mathbf{s}^4(0)$$

The substitutions can also be done in a different order, but the practice of calculation assumes—and it can be formally shown—that the result is always the same, viz. $\mathbf{ssss}(0)$. This term consists entirely of primitive vocabulary, hence there are no more definitions to unfold, and the computation halts.

As a model of computation, definitional unfolding has the virtue of operating directly on arithmetical symbolism. To calculate $2 + 2$ on a Turing machine, for instance, one must specify the initial configuration of a certain machine, where the addition function has become a set of instructions and the 2's have become, say, strings of 1's. In definitional unfolding, by contrast, one operates directly on arithmetical terms and makes use of definitions that are anyhow included in the theory, quite independently of their role in computation. Moreover, the translation of a computational task into a Turing machine configuration itself involves steps of computation, namely the translation of the arguments into sequences of 1's. One sees this clearly by considering the sum $(2+2)+2$. In order to calculate this on a Turing machine, one in effect first needs to calculate $2 + 2$ so as to get a sequence of four 1's.

Definitional unfolding is a special case of a term rewriting system, a model of computation that is well known in the literature. Namely, a system of definitional unfolding is a term rewriting system in which the rewrite rules are nominal definitions directed from definiendum to definiens. Combinatory logic and lambda calculus are paradigmatic examples of term rewriting systems. In the second (shorter) part of the talk I will ask whether they are also examples of definitional unfolding.

That combinatory logic is an instance of definitional reduction is easy to see, since the rule that governs a combinator is just its nominal definition. The K-combinator, for instance, is governed by the rule $K(x, y) \to x$, and this has the form of an explicit function definition.

In lambda calculus, the basic rule is as follows:

$$(\lambda x.t)u \to t[u/x] \tag{β}$$

This rule can be seen to have the form of an explicit definition, namely of the variable-binding λ-operator. In a simply typed setting, the λ-operator takes a term t of type N and a variable x of type M, and yields a function, $\lambda x.t$, of type $(M)N$. An object of function type is explained by saying how it operates on arguments. In particular, we explain the function $\lambda x.t$ by saying how it operates on an arbitrary argument u of type M. This is precisely what (β) does. Since (β) thus explains what $\lambda x.t$ is for arbitrary t, it can be regarded as a definition of the λ-operator. Since the definiendum here has the form 'operator followed by arguments', we may, moreover, regard (β) as a nominal definition. Beta conversion—the term rewriting system whose only rewrite rule is (β)—is thus an instance of definitional unfolding.

A set of rules often considered in addition to (β) are the following:

$$\lambda x.t(x) \to t \tag{η-red}$$

$$t \to \lambda x.t(x) \tag{η-exp}$$

Neither of these rules can be considered a definition. It can be argued, however, that both rules preserve so-called definitional identity, since they formalize the practice of defining a higher-order term $f(x_1, \dots, x_m)$ by means of a lower-order definiendum $f(x_1, \dots, x_m, y_1, \dots, y_n)$. The rule ($\eta$-exp), in fact, turns out to be a useful technical supplement to the theory of definitional identity. Whereas two definitionally identical terms of function type may fail to yield the same term upon definitional unfolding, their unfolding will always result in the same term if one also has access to the rule (η-exp).

Reasoning in Commutative Kleene Algebras from *-free Hypotheses

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Iteration, or Kleene star, is one of the most interesting algebraic operations in computer science. Kleene algebras, besides Kleene star, include the join operation, $+$, concatenation, $\cdot$, and constants 0 and 1. A Kleene algebra is an idempotent semiring $(A, +, \cdot, 0, 1)$ with iteration a^*, which is required to obey two least fixpoint conditions simultaneously:

$$a^* = \min_{\preceq}\{b \mid 1 + a \cdot b \preceq b\} = \min_{\preceq}\{b \mid 1 + b \cdot a \preceq b\}.$$

Here $\preceq$ is the natural preorder induced by $+$: $c \preceq d \iff c + d = d$.

An interesting subclass of Kleene algebras consists of *-continuous ones, in which $ba^*c = \sup_{\preceq}\{ba^nc \mid n \geq 0\}$.

Kozen (2002) studied the complexity of reasoning from hypotheses in Kleene algebras, that is, deciding whether statements like $a \preceq b \Rightarrow a^* \preceq b^*$ are generally true or not. More formally, the algorithmic question involved is checking validity of universally quantified quasiequations, or Horn clauses, of the form $(A_1 \preceq B_1 \mathbin{\&} \ldots \mathbin{\&} A_k \preceq B_k) \Rightarrow C \preceq D$, where $A_1, B_1, \ldots, A_k, B_k, C$, and D are terms constructed from variables and constants 0 and 1 using $+$, $\cdot$, and *.

The interesting case is the *-continuous one, where very high levels of complexity can be obtained. Namely, for arbitrary hypotheses the problem is Π_1^1-complete (!), and if one disallows using Kleene star in the hypotheses (such hypotheses are called *-free), then it becomes Π_2^0-complete. For reasoning in arbitrary, not necessarily *-continuous, Kleene algebras, one gets Σ_1^0-completeness, already with *-free hypotheses.

We consider an natural subclass of Kleene algebras, namely, the class of commutative ones (that is, $a \cdot b = b \cdot a$ for any a, b). The importance of this subclass was noticed by Pratt (1991): when reasoning about computations, the commutative $\cdot$ stands for parallel composition. For reasoning from *-free hypotheses in commutative Kleene algebras, we obtain the same complexity estimations, as Kozen does in the non-commutative case. Namely, we prove Π_2^0-completeness in the *-continuous case and Σ_1^0-completeness in the general one.

Our approach is based on encoding Minsky machines (Minsky, 1961), which are more commutative-friendly than Turing ones. The ideas of this encoding come from Lincoln et al. (1992) and Kuznetsov (2020). Each machine has 3 counters, a, b, c, which hold natural values, and its configuration is encoded by a "commutative word" $qa^ib^jc^k$ (q is the state of the machine). Increasing a counter, say a, and changing the state from p to q is encoded by the following hypothesis: $p \preceq q \cdot a$. Decreasing, with a zero-check, is encoded by two hypotheses: $p \cdot a \preceq q_1$ and $p \preceq q_0 + z_a$. Here z_a stands for a specific state for checking that a is zero.

As in Kozen's article, we prove Π_2^0-hardness by encoding the totality problem for Minsky machines: given a machine, determine whether it halts on any input. We suppose that each machine starts at state q_0 (with its input in a) and halts at state q_F. Now totality, for a given machine, is equivalent to the fact that our hypotheses entail the following statement, in all *-continuous commutative Kleene algebras:

$$q_0 \cdot a^* \preceq (q_F \cdot a^* \cdot b^* \cdot c^*) + (z_a \cdot b^* \cdot c^*) + (z_b \cdot a^* \cdot c^*) + (z_c \cdot a^* \cdot b^*).$$

The encoding from computations (totality of Minsky machines) to reasoning in algebras is straightforward. For the more interesting "backwards" implication, unlike Kozen (2002), we use a syntactic approach. Namely, we embed hypothetical reasoning in algebras to a Gentzen-style calculus with a linear logic style exponential modality, utilizing a sort of deduction theorem, see Kanovich et al. (2019). In this calculus with exponential, we perform cut elimination and analyze cut-free derivations.

For the general, not *-continuous case, we actually proceed without Kleene star, and encode halting of a Minsky machine when started on zero input. This is similar to the proof of Lincoln et al. (1992) for propositional linear logic. We include "garbage collecting" rules for q_F, z_a, z_b, and z_c, ensuring that in the end all counters are zero. Then we have to entail

$$q_0 \preceq q_F + z_a + z_b + z_c.$$

This yields Σ^0_1-hardness, even without Kleene star at all. For the upper bound, we just notice that now we can do everything in a calculus with finite proofs, which gives recursive enumerability.

An important thing to notice is that in our encoding we essentially need $+$ in hypotheses. In the non-commutative situation, already monoid equations, that is, hypotheses of the form $u = v$, where u and v are built using only $\cdot$, are sufficient. This is the same phenomenon as in propositional linear logic (Lincoln et al., 1992). For reasoning in commutative *-continuous Kleene algebras from sets of hypotheses which are commutative monoid equations, we leave the question open, and conjecture that it could be decidable. We also conjecture Π^1_1-completeness for the case of unrestricted hypotheses.

This work is supported by the Russian Science Foundation under grant 21-11-00318.

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Hyperdoctrine Semantics: An Invitation

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Categorial logic, as its name suggests, applies the techniques and machinery of category theory to topics traditionally classified as part of logic. For certain philosophers—mainly mathematical logicians—the small taste of categorial logic offered in this paper will be nothing new. What we claim and intend this paper to demonstrate, however, is that these tools deserve attention from a greater range of philosophers than just the mathematical logicians.

We support this central claim with an example. In this paper we show how to use one tool from categorial logic—hyperdoctrines—to do interesting metaphysics. Hyperdoctrines, among other things, can provide semantics for quantified languages. But the metaphysical picture suggested by hyperdoctrines is in many respects quite different from the metaphysical picture suggested by the model-theoretic semantics of the quantifiers.

In an ordinary model (of the signature Σ, say) one has first of all, a domain of individuals. One then has the interpretations of the various symbols of Σ: interpretations of predicates are sets of individuals, interpretations of constants are individuals, and so on. Thus, the whole model-theoretic edifice *grounds out*, in some sense, at the level of individuals.

The grounding going on here, whatever it might be, is a fairly robust matter. To begin, there is the obvious dependence of all of the sets figuring in the interpretation on their members. But even putting that to the side, the comparison of structures ultimately comes down to the sets of individuals in their domains; morphisms of Σ-structures are simply maps between the underlying domains of the respective structures that happen to have certain further properties.[1] The grounding of the model-theoretic world on the world of individuals and particulars further reveals itself on even a casual examination of many of the classical results of the subject. As often as not, said results are either statements about possible cardinalities for structures, or statements about how many structures there are (up to isomorphism) of a certain cardinality. Making generalizations about the psychology of workers in a scientific field is a risky business, but it seems fair enough to say that the models are fundamentally understood to be decorated sets (like groups, fields, and other objects in concrete categories), and that their underlying sets and the individuals that inhabit them are fundamental to the subject.

This incursion of set theoretic concepts into metaphysics, where sets are smuggled in as indispensable for model-theoretic semantics, has deeply colored contemporary analytic philosophy, both subtly and overtly. Tim Maudlin's criticism of what he calls "set theoretic extensionalism", and more generally the Quinean approach to ontology puts the point nicely. We have, in the analytic tradition:

> . . . a fundamental metaphysical picture: in the world, there are objects, which are referred to by singular terms and quantified over by first-order variables, and

[1] Categorically speaking, what this amounts to is that models in any signature are implicitly a concrete category: they come equipped with a canonical "forgetful" functor (the functor induced by the underlying set-theoretic structure of the models) targeting Set.

> there is some structure to the set of objects provided by universals... There is, of course, something suspicious in the way the subject/predicate structure of the language seems to be mirrored in the ontology, but the question remains: what is the alternative? (Maudlin, 2007, p86)

For further evidence, consider, for example, the influential slogan “to be is to be the value of a [bound] variable” from Quine (1948), the problem of absolute generality as discussed in Parsons (2006), or Putnam’s model-theoretic argument for anti-realism Putnam (1980).

The novelty of our presentation is this: we present a respectable semantics for quantified first-order logic without any appeal to things that are being quantified over. We have, if you like, Being ($\exists$, that is) without beings. Insofar as hyperdoctrine semantics witnesses that the model-theoretic picture is not at all mandatory for a natural treatment of quantification that (a) is philosophically fruitful and (b) has bearing on debates in metaphysics, it has a role to play in broadening the space of philosophical possibilities.

This broadening of the space of philosophical options is interesting on its own. But there’s more to the value of hyperdoctrine semantics than this. The particular moral we’ll focus on here is the following: unlike the picture we get from the model-theoretic paradigm, in the hyperdoctrinal framework, adopting nonclassical logic does not require commitment to anything *intensional*.

Let’s make clear what we mean: on the model-theoretic paradigm, many interesting non-classical logics *only* have semantic theories that interpret one or more connectives *intensionally*. For example, semantic theories for logics in the relevant family all feature either sets of worlds, or sets of theories, or sets of setups, or sets of information states or sets of contents that are linked up by one or more pieces of machinery (binary operations, ternary relations, etc.). And, in these theories, the conditional (and often the quantifiers, when they are present) are interpreted *intensionally*—they are evaluated at a particular point not using only the information at that particular point, but also information found at other points in the model.

Much the same is true for logics in the intuitionistic and dual-intuitionistic and linear families. This commitment to intensionality seems to be, to many authors, an important point. Those with a Quinean bent take it to be problematic and reason to rule out such logics. Those who are more friendly to intensionality might take it to be a point in favor of these logics and against classical logic. Regardless, it’s often taken to be significant.

In hyperdoctrine semantics, intentionality is not inevitable for the families of non-classical logics enumerated above. Regardless of the logic one adopts, hyperodoctrine semantics is interpreted in terms of propositions hooked up in roughly the same ways—the structure of a classical first-order model is not a privileged case. The commitment to propositions might be taken to entail be a commitment to intensionality across the board (propositions sure *seem* to be intensional in flavor) or a commitment to extensionality across the board (there are no worlds/theories/whathaveyou in sight, after all). The crucial point is that the apparent correlation between the non-classical and the intensional is revealed, on the perspective we present, to be merely illusory.

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A Formal Perspective on Purity of Proof: When Does a Mathematical Proof Belong to the Content of a Theorem?

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Purity has a long history as an ideal of proof for mathematicians, tracing back to early writings of Aristotle and Archimedes (Detlefsen, 2008). It concerns the idea that a pure proof should only draw upon notions that belong to the content of the theorem. Impure proofs distinguish themselves from pure proofs by making use of concepts that are 'extraneous' to what a theorem is about. A traditional value of purity is that it allows us to "become familiar with the specific details of the subject of the theorem" (Lehet, 2021), while impurity is commonly valued for its ability to unify and generalize different disciplines of mathematics. Purity and impurity have also been related to other conceptual values of proof such as simplicity and explanatoriness (Arana, 2017; Iemhoff, 2017; Lange, 2019).

We will propose a new understanding of purity of proof, that better accommodates the attitude of contemporary mathematics by preserving values of both traditional purity and impurity. Our approach also contributes to characterizing how purity can be incorporated in a proof-theoretic setting. This will help our understanding of how criteria for formal derivations can correspond to philosophically expressive and meaningful properties — a topic that has not yet enjoyed many substantial results. For example, it has proven difficult to provide satisfying proof criteria that correspond to the informal property of simplicity (referred to as Hilbert's 24$^{\text{th}}$ problem) (Hipolito & Kahle, 2019). As for purity, previous accounts generally aim to explain its practical manifestation (Arana & Detlefsen, 2011; Baldwin, 2013; Kahle & Pulcini, 2017). Resulting frameworks usually leave room for mathematicians to translate their intuitions into more delineated concepts. Instead, Arana (2009) investigates the use of cut elimination as a property mechanically guaranteeing purity for formal proofs, but concludes that this does not accurately represent practical purity. We will provide some additional reasons for why this measure is imperfect, and focus on an approach that incorporates the intuitions of mathematicians. In fact, we will argue that our approach additionally allows us to *extend* the intuitions of mathematicians concerning purity, once we are in a formal setting.

First, we take a new perspective on what a mathematical theorem is about. We interpret the content of a theorem as the range of mathematical material that a theorem concerns, as captured by a particular formal theory. We argue that it is reasonable to think that this is what underlies mathematician's intuitions when they make purity statements in practice. This perspective ensures that purity is brought into a formal setting (by the crystallization of content into a formal theory), while its intuitive nature is preserved (by letting the selection of this theory be heavily inspired by mathematical intuitions). Syntactic derivations that start from the axioms of the pure formal theory may then be considered pure. In order to make this purity guarantee as inclusive as possible with respect to the intuitions of mathematicians, however, a notion of equality for theories is desirable. That is, we want the purity condition to incorporate formal theories that only differ from the pure theory in superficial ways, and that still represent the same mathematical intuition. Given that the mathematical community considers a formal theory pure for a theorem, we suggest that it is reasonable to consider all definitional extensions of this theory as equal choices for purity.

Second, we argue that we can extend purity intuitions by using interpretations. This will concern the formal proofs of theories that do not directly capture the intuitions of mathematicians for a theorem. However, such theories may certainly be able to express and prove all (translated) theorems of the pure formal theory. Given some requirements that ensure accuracy of a translation, translated proofs may be said to 'simulate' original (pure) proofs. The notion of interpretation between theories (see, e.g. (Visser, 1997)) has suitable ingredients that allow for a proof-theoretic condition for proof simulation, which comes down to the requirement on formal proofs that each branch begins with the derivation of an interpreted pure axiom. Since such simulating formal proofs remain understood from a set of potentially impure axioms, purity cannot strictly be satisfied. Still, we argue that close simulations of pure proofs should be attributed a sense of purity, as they reduce a theory to proving all and only (translated) properties of the pure theory. In other words, no extraneous notions of the possibly impure theory can creep in, and we might even say that restricting to simulations restrains the use of primitive notions of the interpreting theory in such a way that their meaning approaches that of components of the pure theory.

Thus, by capturing intuitions about purity with formal theories and extending them with interpretations, we form a first understanding of how to consider purity of formal proofs. While the problem of finding a fully mechanical method that determines intuitive purity results remains open, we point out that informal conceptions can fruitfully go together with formal tools, in order to refine intuitions as well as bridge the gap between informal and formal concepts.

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Two Faces of Logical Minimalism

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In this paper, we propose to clarify the notion of Logical Minimalism. We will accomplish this by investigating whether the non-reflective logic proposed by Fjellstad (2020) has useful laws to do metatheory.

Logical Nihilism is the thesis that there are no laws of logic (Russell, 2018 2017). Logical Minimalism is the thesis that there are just few logical laws or, equivalently, *there are hardly any logical laws* (2017 & 2018) and those that exist are not useful for doing metatheory. As support for Logical Nihilism, Pailos (202X) presents an "Empty Logic", a logic that has no laws of logic. On the other hand, Dicher (2020) and Fjellstad (2020) raised very interesting criticisms on Logical Nihilism as characterized by Russell. Dicher makes some criticisms as that the empty consequence is not as easy to achieve as Russell believes, and attacks the criteria for cataloging a set of arguments as minimalist. On the same line, Fjellstad (2015) defends that a set of arguments of a non-reflexive logic does not necessarily lead to Nihilism nor Minimalism.

Our hypothesis is that it has not yet been determined that Fjellstad's non-reflective logic has useful arguments and conditions to do metatheory, and therefore that it is not logical minimalist. Thus, our objective is to determine whether Fjellstad's logic really meets all the conditions so that the set of arguments cannot be considered as logical minimalist. The importance of this research is that it broadens the understanding of logical nihilism and logical minimalism, and the relationship between them. We will reflect on the set of criteria and arguments that are useful to do metatheory, and finally, we will propose a characterization of *logical minimalism* that captures these characteristics.

We assume a propositional language $\mathcal{L}$ that has the set of connectives $\{\wedge, \vee, \sim, \rightarrow\}$. The formulas built from this set of connectives are interpreted as in **FDE**. The relation of entailment $\Gamma \vdash \Delta$ is defined as usual. We call 'Γ' the set of premises and 'Δ' the set of conclusions. We use the notation $\mathcal{L}_{\{\circ\}}$ to denote the expansion of the set of connectives of $\mathcal{L}$, as follows: $\{\wedge, \vee, \sim, \rightarrow, \circ\}$. We also consider the corresponding definition of formulas restricted to this language. In particular, we will consider the expansion of $\mathcal{L}$ with the nullary $\ddagger$ and the unary $\neg$, that have the following truth and falsity conditions:

2

- $1 \in i(\neg A)$ if and only if $1 \notin i(A)$
- $0 \in i(\neg A)$ if and only if $0 \notin i(A)$.
- $1 \in i(\ddagger)$ if and only if $\ddagger \in \Gamma$
- $0 \in i(\ddagger)$ if and only if $\ddagger \in \Delta$

A reflexive logic is a logic that validates $A \vdash A$. Employing $\ddagger$, Russell obtains a counterexample to identity: $\ddagger \vdash \ddagger$. Consequently any logic with $\ddagger$ is a non-reflexive logic.

We also consider the expansion of $\mathcal{L}$ with the binary $\to_f$ with the following non-truth condition:

- $1 \notin i(A \to_f B)$ if and only if $1 \in i(A)$ and $0 \in i(B)$

Fjellstad's argument is then the following one:

(F0) **Bad Face of Logical Minimalism**: a non-empty set of valid arguments is minimalist if it does not have any argument useful for doing metatheory or assessing proofs in arithmetic.[1]

(F1) In the non-reflexive logic based on the language $\mathcal{L}_{\{\ddagger, \neg\}}$, one has Modus Tollens as a valid argument.

(F2) In the non-reflexive logic based on the language $\mathcal{L}_{\{\ddagger, \to f\}}$, one has Modus Ponens as a valid argument.

(F3) If one can have inferences like Modus Tollens or Modus Ponens, then one has valid inferences that are useful for doing metatheory.

(F4) In the non-reflexive logic based either on $\mathcal{L}_{\{\ddagger, \to f\}}$ or $\mathcal{L}_{\{\ddagger, \neg\}}$, one can have a inference useful for doing metatheory.

(F5) The set of inferences built from a set of connectives $\mathcal{L}_{\{\ddagger, \neg, \to f\}}$ is not minimalist.

As a corollary, a set of inferences of a non-reflexive logic is not (necessarily) minimalist.

We believe that F5 is true. However, we consider that the proof of Fjellstad is incomplete and in order for us to affirm that there is a non-reflexive logic (with the ‡ in place) that is not minimalist, theoretical development is needed on at least two fronts:

OF1 It is necessary to clarify what is meant by logical minimalism. Without clarity about the conditions that a set of arguments has to satisfy to do metatheory, it is impossible to understand what Logical Minimalism really is. In this sense, we propose to give a definition of Logical Minimalism that is compatible with a set of sufficient conditions to do metatheory (we will call this definition **Good Face of Logical Minimalism**). We will argue that our definition is better than Fjellstad's since it allows taking into account other arguments, in addition to Modus Ponens, that every set of arguments must have in order not to be logical minimalist.

OF2 After having a definition of logical minimalism with a set of sufficient conditions to do metatheory, we will give a list of inferences that can be considered as sufficient for a set of inferences not to be minimalist. This set of valid inferences to do metatheory is built from the minimum requirements that the negation and conditional connectives must have in order to develop arithmetic tests.

Finally, we conclude our paper with the respective tests about whether Fjellstad's non-reflective logic satisfy the requirements to avoid logical minimalism.

[1] Actually, Fjellstad refers in this premise to Logical Nihilism. The confusion between Logical Nihilism and Logical Minimalism could be originated from the fact that Russell herself put both of them on the same level, in that "everything that seems bad about the one seems bad about the other."(Russell , 2018, p.15)

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What Speech Act is Tied to a Contradiction?

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The notion of absurdity or contradiction, commonly denoted by $\perp$, has three key appearances in modern logic, specifically in natural deduction for intuitionistic and classical logic. The first one is in the reductio ad absurdum rule, arguably one of the most important proof techniques, which can be schematized as follows:

$$\frac{\begin{array}{c}[A]\\ \perp\end{array}}{\neg A}$$

and read as "if absurdity follows from A, then we can derive $\neg A$". The second appearance is in the definition of negation, another crucial logical notion, which proceeds as follows:

$$\neg A =_{df} A \supset \perp$$

and states that negation of A can be reduced to implication of absurdity. In other words, asserting "A is not true" can be understood as equivalent to asserting "A implies absurdity". The third appearance is in the ex falso quodlibet rule, also known as the principle of explosion:

$$\frac{\perp}{A}$$

that captures the idea that anything follows from absurdity.

But despite the undeniable significance of $\perp$ there are still unresolved questions in logical literature about its precise nature. What is $\perp$? Is it a proposition? Specifically the definite "false proposition" as originally suggested by (Gentzen, 1969, p. 70), and thus not a logical connective?[1] Or is it a logical connective, specifically the zero-place logical connective (= propositional constant, nullary operator), either with no introduction and elimination rules, as initially considered by (Prawitz, 1965, p. 19), or with only elimination rule, as initially considered by (Martin-Löf, 1971, p. 189), which since then became arguably the most common treatment of $\perp$ in both textbooks and research papers?[2] Or maybe is it nothing at all, just a structural punctuation mark informing us that a derivation reached a "logical dead-end", as proposed by (Tennant, 1999, p. 205)?[3]

Then there are questions connected with its interpretation: What does $\perp$ mean? Different authors prefer different names, the three most common are absurdity, contradiction, and falsity.[4] But more importantly: Where does $\perp$ get its meaning from? Can it be specified by rules of inference or do we have to take it as a primitive notion? This question seems of a special relevance to proof-theoretical and inferential approaches to semantics.

To these considerations is connected another important question: Why is $\perp$ considered bad? Is it bad in itself or is it bad because it "indicates that there is something wrong about

[1] In German: "die Falsche Aussage", (Gentzen, 1935, p. 178)

[2] See, e.g., (Troelstra & van Dalen, 1988), (Dummett, 1991), (Buss, 1998), (Troelstra & Schwichtenberg, 2000), (Negri et al. , 2001), (Mares, 2011), (von Plato, 2014), (Rumfitt, 2017).

[3] See also (Rumfitt, 2000), (Steinberger, 2011), (Murzi, 2020).

[4] With possible variations: falsehood, falsum, False.

our calculus" and that "it is merely the (local) *symptom* of a sickness of the whole body," as mentioned by (Wittgenstein, 1956, II-81; 104e)?

In this paper, I will argue that many of the disagreements around absurdity stem from the fact that the corresponding symbol $\bot$ is simply overloaded. It seems to play two different roles: $\bot$ as a proposition (allowing us to define negation) and $\bot$ as a structural punctuation mark (informing us that we have reached a logical dead-end). Moreover, I will argue that both these roles are indispensable. The traditional meaning of $\bot$ cannot be exhausted by simply treating it only as a proposition, or only as a punctuation mark, it seems that we need both readings. Thus getting rid of one reading and keeping only the other does not seem as a viable strategy of solving the ambiguity of $\bot$.

Furthermore, I will argue that $\bot$ on the second reading is not just any punctuation mark but that it plays the role of an exclamation mark indicating a change of illocutionary force from assertoric to imperative. In other words, I will treat the symbol $\bot$ as an imperative force indicator, analogously to the symbol $\vdash$ which plays the role of an assertoric force indicator. Consequently, I will approach absurdity not as a proposition or an assertion of a proposition but as a command (specifically, the impossible command). By adopting this view, we can make sense of various issues concerning $\bot$.

Thus, as a solution to some of the disagreements surrounding $\bot$, I propose unwinding it into two separate notions: for the first role (i.e., reading $\bot$ as a proposition, specifically, an empty proposition that cannot be proven) I will use the symbol $\varnothing$ and for the second role (i.e., reading $\bot$ as a force indicator, specifically the imperative force indicator) I will keep the symbol $\bot$. The standard natural deduction rules associated with $\bot$, i.e., negation introduction (= reductio ad absurdum), negation elimination (= ex contradictione falsum), and the absurdity rule (= ex falso quodlibet) will then become as follows (explicitly displayed forces indicate the appropriate speech acts):

$$\frac{\begin{array}{c}\dashv A\\ \bot\ \varnothing\end{array}}{\vdash A \supset \varnothing} \qquad \frac{\vdash A \qquad \vdash A \supset \varnothing}{\bot\ \varnothing} \qquad \frac{\bot\ \varnothing}{\vdash A}$$

where $\dashv$ (following Sundholm, 2006) denotes an "assumptory" force. How do we interpret $\bot\ \varnothing$? It is a command to do $\varnothing$, or more precisely, a command to make $\varnothing$ true. And since $\varnothing$ cannot be true by definition (it is the empty/false proposition), $\bot\ \varnothing$ then effectively becomes an impossible command, i.e., a command that cannot be fulfilled. And, arguably, the most natural effect of such a command is nothing at all – it would abort or *jam* (to borrow a term from Wittgenstein, 1975) the derivation.

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Truthmaker Semantics for Containment and Nonsense

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Truthmaker semantics (Fine, 2017) is a hyperintentional theory of truth-conditional content. It is a generalisation of possible-world semantics in which possible worlds are replaced by possible but also impossible partial states. In the last decade, it has attracted a growing attention and has been applied to various subfields of philosophical logic. In particular, Fine's framework provides a semantics for the Strong Kleene logic $\mathtt{K}_3$ and the Belnap-Dunn logic $\mathtt{FDE}$ (see Priest (2008)).

In this talk, I put forward a motivated modification of Fine's semantics concerning the treatment of disjunction. After this modification, Fine's semantics for $\mathtt{K}_3$ becomes a semantics for the Weak Kleene logic $\mathtt{K}_3^w$ (Priest, 2008) and his semantics for $\mathtt{FDE}$ becomes a semantics for Oller's logic $\mathtt{AL}$ (Oller, 1999).

One advantage of the proposed truthmaker semantics is that it is easily generalised to other propositional languages. More precisely, any algebra $\mathcal{A}$ of any type gives rise to a truthmaker semantics that I call $\mathcal{A}$-TM. The modification of Fine's framework that I describe corresponds to $\mathbf{2}$-TM, where $\mathbf{2}$ is the Boolean algebra defining classical logic.

These truthmaker semantics qre closely linked to plurivalent semantics. Plurivalent semantics is a very flexible framework developed by Priest (2017) in which one can construct semantics for certain subsystems of any many-valued logic. For instance, $\mathtt{K}_3^w$ corresponds to the singular plurivalent logic (Szmuc and Omori, 2018) induced by $\mathbf{2}$. Similarly, $\mathtt{AL}$ is the general plurivalent logic induced by $\mathbf{2}$. Generalising from the semantics for $\mathtt{K}_3^w$ and $\mathtt{AL}$ in $\mathbf{2}$-TM, I show that, for any algebra $\mathcal{A}$, the framework $\mathcal{A}$-TM contains a semantics for the singular plurivalent logic and a semantics for the general plurivalent logic induced by $\mathcal{A}$. Since singular plurivalent logics corresponds to nonsense logics (or infectious logics) (Ferguson, 2015), I thereby give a sound and complete truthmaker semantics for all nonsense logics.

To further illustrate the usefulness of the link between these truthmaker semantics and plurivalent semantics, I develop a new type of plurivalent semantics for containment logics (Ferguson, 2015), namely logics defined by imposing a syntactic containment condition on top of another logic. More precisely, an entailment from φ to ψ holds in the containment logic induced by a logic if that entailment holds in that logic and every propositional variable of ψ occurs in φ. The plurivalent understanding of containment is used to develop a truthmaker semantics for containment logics. I use this purely semantic characterisation of containment logics to challenge an objection by Routley (Routley et al. (1982)), according to which they are mere syntactic artefacts without semantic depth.

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Unsharp Quantum Measurements as a States of Knowledge: Epistemic Logic of Quantum Information

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In 1932, John von Neumann wrote his famous work "Mathematical Principles of Quantum Mechanics", which determined the direction of development of quantum physics. It presented the mathematical and logical foundations of quantum mechanics, in particular, presented a sketch of the idea of *quantum logic*. This idea was co-authored by von Neumann with G. Birkhoff in his work "The Logic of Quantum Mechanics" (Birkhoff, Von Neumann, 1936), from which quantum logic begins as an independent section of non-classical logic.

The main content of the logic presented by Neumann is reduced to the analysis of the lattice of projection operators formed by the experimental statement, which later named as an *orthomodular lattice*. Experimental statements are established as a statements that an arbitrary observable has a definite meaning. Any experimental statement is associated with a subset of Hilbert space, which contains all of its pure states - the most complete descriptions of a quantum system. Von Neumann and G. Birkhoff called such subsets of physical properties: any quantum physical system a certain set of characteristics that can be investigated as a result of measuring such a system. Thus, as a first approximation, it turns out that experimental statements are identical with the information that one can have about a physical system. Moreover, this information shows the lattice of statements does not have a probabilistic nature, as is typical for quantum physics. The main problem of von Neumann's approach was to study only *pure states* as objects of quantum reasoning. This problem finds its solution in an *unsharp approach*.

The *unsharp approach* to quantum theory was firstly proposed in (Ludwig, 1983). Proponents of this approach accused orthomodular quantum logic systems of *totality* and *sharpness*. The totality meant that the statements of quantum logic were necessarily closed with respect to the conjunction operation, and the sharpness meant that, in the standard interpretation, the statements corresponded to the exact possible properties of the quantum systems under study. This was due to the fact that the values of the observable data obtained as a result of the measurement lay in a well-defined numerical range.

One of the basic ideas of the unsharp approach was the exploitation of the linear bounded operators instead of projection operators which were a main tool of the sharp approach to quantum theory. These operators has been called *effects* in the framework of unsharp investigations. The main difference between the projections and proper effects from this point of view intuitively should be described as the diversity of answering the question "do value for the observable A lies in given Borel set?" In case of projections we should addressed to *exact* Borel set while for effects our Borel set should be *fuzzy*. Unlike the orthomodular lattice, which is usually introduced on the set of all projectors $P(\mathcal{H})$, various algebraic structures can be induced on the class of all effects $E(\mathcal{H})$ (see Chiara, Giuntini, Greechie, 2004). In particular, the most promising for the study of unsharp quantum logic is the algebraic structure *effect algebra* (Foulis, Bennett, 1994) , which turned out to be the most general among the structures describing the structures of quantum states, or rather, dealing with the description and transformation of information about quantum systems. It is important to note that effect algebras are closely related to *weak (unsharp) measurements*, which assume a *continuous* quantum measurement that would not violate the integrity of the system under study.

The relationship between effect and state can be described as the probability with which the object under study has the intended property. This approach to the construction of the theory of measurements, together with modern technical capabilities for carrying out measurements (POM-measurements), allows us to say that quantum systems are being investigated as *epistemic objects*. We can say that pure states correspond to complete and reliable knowledge about an object in a quantum system, and mixed states can be defined as a state of uncertainty about one's own knowledge. In this case, the degree of uncertainty will be characterized by a probabilistic value that obeys the Born rule. The meaning of the latter is that the sum of the squares of the probability amplitudes characterizing each of the eigenstates is equal to **1**. Considering that, in the general case, researchers are interested in the presence or absence of some mutually exclusive properties that are identified with two eigenstates, it can be argued that we can talk about a complete understanding of our own knowledge. The application of these structures looks promising in relation to the theory of quantum information, in the spirit of works of A. Baltag and S. Smets (see Baltag, Smets, 2012).

We use the original Kripkean semantics with a ternary relation, constructed similarly to (Vasyukov, 2004), in which developed the idea that quantum logic is a substructural logic, close connected with the linear and relevant logic (Restall, 2000). This approach was chosen in connection with the fact that the ternary relation is more general than binary relation and has greater expressive possibilities. This semantics corresponds to the *algebra of effects* (Foulis, Bennett, 1994). In this context, as a non-empty set of states in a Hilbert space H is taken, where the *state* is defined as the mapping $s : A \mapsto [0, 1]$.

Quantum frame is a structure $\mathcal{F} = \langle \mathcal{S}, \mathcal{R}, * \rangle$, where $\mathcal{S}$ is a non-empty set of states in a Hilbert space H; $\mathcal{R}$ is a ternary relation on $\mathcal{S}$ ($\mathcal{R} \subseteq \mathcal{S}^3$); $*$ is an unary operation on $\mathcal{S}$, mapping states to other states, which incompatible with the original ones ($* : \mathcal{S} \to \mathcal{S}$).

The following definitions are accepted:

$DF1. a \leq b \Leftrightarrow \exists x(\mathcal{R}axb)$

$DF2. a \perp b \Leftrightarrow \exists x(\mathcal{R}abx)$

$DF3. R^2abcd \Leftrightarrow \exists x(\mathcal{R}abx \& \mathcal{R}xcd)$

$DF4. R^2a(bc)d \Leftrightarrow \exists x(\mathcal{R}axd \& \mathcal{R}bcx)$

$\mathcal{R}$ is an **orthosum** relation iff for all a, b, c in X the following postulates are satisfied:

$p1. \mathcal{R}abc \Rightarrow \mathcal{R}bac$

$p2. \mathcal{R}^2abcd \Rightarrow \mathcal{R}^2a(bc)d$

$p3. \forall a \exists! a^*(\mathcal{R}aa^*\mathbf{1})$ (the orthosupplement to a)

$p4. \forall a(\mathcal{R}a\mathbf{0}a)$

$p5. \mathbf{0} \perp \mathbf{1}$

$p6. a \leq b \Rightarrow b^* \leq a^*$

$p7. a^{**} = a$

$p8. \mathbf{1}^* = \mathbf{0}$ and $\mathbf{0}^* = \mathbf{1}$

Quantum model is a structure $\mathcal{M} = \langle \mathcal{S}, \mathcal{R}, *, \rho, \Pi, \mathbf{v} \rangle$, where $\langle \mathcal{S}, \mathcal{R}, * \rangle$ is a quantum frame, ρ is a *verification function* $\rho : \mathcal{A} \times \mathcal{S} \to [0, 1]$, which assigns to any effect in some state its Born probability; Π is a *set of proposition* of the frame, that contains $\oslash, \mathcal{S}$ and closed under orthocomplement $'$ and set-theoretic intersection $\cap$; $\mathbf{v}$ is a function that associates to any sentence α a proposition in Π.

This relational semantics characterises the same logic as the effects algebra. The proposed relational semantics is a semantics that potentially characterise *epistemic quantum logic* that generalising an unsharp approach of quantum measurements. The main idea of development of this structure is the construction of syntax and its axiomatisation in the spirit of epistemic logic, where we use defines bi-modal epistemic operators, which describes quantum information processes and operations on it.

The study was implemented in the framework of the Basic Research Program at the National Research University Higher School of Economics (HSE University) in 2021.

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Decidability and Finitary Character of Proof-Theoretic Validity

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Proof-theoretic validity has proven a useful tool for proof-theoretic semantics. It offers an explanation of the harmony found in the introduction and elimination rules for the intuitionistic calculus. The consequence relation implicit in Prawitz's (1971) original presentation of the notion is that $\Gamma \vDash_{Pr} \varphi$ holds whenever there is a proof-theoretically valid proof from assumptions in Γ to φ. So defined the consequence relation has the property of being finitary, i.e. whenever $\Gamma \vDash_{Pr} \varphi$ there is a finite Δ subset of Γ such that $\Delta \vDash_{Pr} \varphi$. In the work of Piecha, Schroeder-Heister, and Campos Sanz (2015) they uses a consequence relation $\vDash_{PSC}$ to prove impressive results about how proof-theoretic validity works. However, their consequence relation is not finitary on some treatments of the atomic formulas. In this talk, I will show that they do coincide on a broad range of treatments including the most philosophically important ones. This is done by show that the consequence relation is decidable.

Consider the following inference rules that only contain atomic formulas:

$$\text{level 0: } \overline{p} \qquad \text{level 1: } \frac{p}{q} \qquad \text{level 2: } \begin{array}{c}[p]\\ \vdots\\ \dfrac{q}{r}\end{array} \qquad \text{level 3:}^{1} \begin{array}{c}\left[\begin{array}{c}\vdots\\ \dfrac{p}{q}\\ \vdots\end{array}\right]\\ \dfrac{q}{r}\end{array}$$

Proof-theoretic validity is defined relative to the set of all inference rules that contain only atomic formulas, $\mathbb{S}$, and a selection of subsets of this set $\mathfrak{S} \subseteq \mathscr{P}(\mathbb{S})$ called a proof-theoretic system. You can think of $\mathfrak{S}$ as an intuitionistic Kripke model and each set $S \subseteq \mathbb{S}$ from $\mathfrak{S}$ as a world in the model. The accessibility relation is the subset relation and the atomic formulas forced at S are those that can be proven using the inference rules it contains.

The only difference between $\vDash_{Pr}$ and $\vDash_{PSC}$ is that $\vDash_{PSC}$ is "monotone" in that relative to a $\mathfrak{S}$ and S, we have $\Gamma \vDash_{PSC} \varphi$ iff

$$[\forall S' \supseteq S(S' \in \mathfrak{S} \text{ and } \mathfrak{S}, S' \vDash_{PSC} \Gamma \Rightarrow \mathfrak{S}, S' \vDash_{PSC} \varphi)].$$

Where as, Prawitz's notion is finitary as relative to a $\mathfrak{S}$ and S, we have $\Gamma \vDash_{Pr} \varphi$ iff

$$\exists \text{finite} \Delta \subseteq \Gamma[\forall S' \supseteq S(S' \in \mathfrak{S} \text{ and } \mathfrak{S}, S' \vDash_{Pr} \Delta \Rightarrow \mathfrak{S}, S' \vDash_{Pr} \varphi)].$$

It is not the case that every treatment of the atomic formulas leads to these two notions corresponding. We can give toy example on which $\vDash_{PSC}$ is not finitary.

Lemma 1. Let $\mathfrak{S} = \{\{p_0, \ldots, p_n\} \mid n \in \mathbb{N}\} \cup \{\{q, p_0, p_1, \ldots\}\}$ then $\mathfrak{S}, \{p_0, p_1, \ldots\} \vDash_{PSC} q$ but $\mathfrak{S}, \{p_0, p_1, \ldots\} \nvDash_{Pr} q$.

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[1]This rule discharges another rule. It can be thought of instead as the rule $\begin{array}{c}[p \to q]\\ \vdots\\ \dfrac{q}{r}\end{array}$ which discharges the assumption $p \to q$. What is important is the disjunction does not occur in the rules.

$$
\begin{tikzcd}
& \{\bar{\bot}\} & \\
\{\bar{p}, \frac{\bot}{p}\} \arrow[ur] & & \{\frac{p}{\bot}, \frac{\bot}{p}\} \arrow[ul] \\
& \{\frac{\bot}{p}\} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Figure 1:

But some $\mathfrak{S}$ are more natural and popular than others. We can ask whether $\vDash_{Pr}$ and $\vDash_{PSC}$ coincide on the proof-theoretic systems: $\mathfrak{S}_n = \mathscr{P}(\{R \mid R \text{ is a level-n or below rule}\})$ and $\mathfrak{S}_\infty = \mathscr{P}(\mathbb{S})$. The goal of this talk is to show that they do in fact correspond on these systems. This will be done by showing that these systems are decidable.

This is easily shown for $\mathfrak{S}_\infty$ which is known to correspond to generalised inquisitive logic (Stafford forth; Punčochář 2015). However, it is more difficult for the $\mathfrak{S}_n$. The proof proceeds by first showing that

$$\mathfrak{S}_n, S \vDash_{PSC} \varphi \Leftrightarrow \mathfrak{S}_n^{[p_1,\dots,p_n]}, S \vDash_{PSC} \varphi$$

where $p_1,\dots,p_n$ are the atomic variables in φ and $\mathfrak{S}_n^{[p_1,\dots,p_n]}$ is $\mathfrak{S}_n$ restricted to atomic rules that only contain $p_1,\dots,p_n$.

The next step involves showing that $\mathfrak{S}_n^{[p_1,\dots,p_n]}$ can be further restricted to a finite collection of finite sets of rules. From that decidability follows via simply checking φ at each $S \in \mathfrak{S}_n$. This can be illustrated by considering an example. Figure 1 displays the reduced system for $\mathfrak{S}_1^{[p]}$. That is the system containing only rules of level 1 or lower and only the atomic variable p. We can now easily check the known result that $\mathfrak{S}_n \vDash_{PSC} \neg\neg p \to p$ by checking each of the four cases.

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Model-Theoretic and Proof-Theoretic Analyses of Deontic Modality: Agentive and Linguistic Rejection

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In the last few years, two lines of research on expressivist interpretations for deontic modal operators have been advanced, one in proof theory and the other in model theory. This talk compares these two interpretations, and examines the prospect of their unification.

Building off of Smiley (1996), Rumfitt (2000), and Restall (2005), the proof-theoretic approach has been developed by Luca Incurvati and Julian Schlöder (2017, 2019, Forthcoming). They start with a bilateral natural deduction system employing introduction and elimination rules for speech acts of assertion and rejection, which count as expressing attitudes of assent and dissent. These rules treat assertion and rejection as unembeddable attitude markers. They generalize this bilateral semantics to a trilateral semantics by adding an attitude of weak assent, expressed by speech acts of weak assertion, so as to give introduction and elimination rules for epistemic modals. In Incurvati and Schlöder (Forthcoming), attitudes of disapproval and approval are used to supply rules for using deontic modal vocabulary. This semantics for the deontic modalities counts as *expressivist* insofar as the assertion of a claim like 'you ought to be polite to your neighbors' is understood as expressing *approval* of doing so.

Bilateral semantic programs allow for the systematic reduction of the assertion of logically complex sentences to expressions of commitment to assent and dissent of atomic sentences; e.g., the assertion of a negation is understood as a rejection of the negate, and the assertion of a conditional is understood as expressing commitment to avoiding both asserting the antecedent and rejecting the consequent. In this way, grasp of logically complex propositional content can be understood in terms of one's capacity to obey various rules regarding what to assert and deny of the nonlogical atoms of a language. This fact in turn supports inferentialist programs, like those of Brandom and Peregrin, that understand human cognition in terms of our ability to reason about the world. For proof-theoretic methods like those employed in bilateral semantic programs are ripe for reconstructing inferentialist orders of explanation, foregrounding notions of meaning that are intralinguistic and involve relations of proper and improper inference within language.

Model-theoretic notions of meaning, by contrast, are better fit for reconstructing those dimensions of meaning that involve relations between language and the world, whether in the language-to-fit-world intentionality of theoretical cognition, or the world-to-fit-language intentionality of practical cognition. Intriguingly, this view sees proof-theory as analogous to the central moment of the reflex arc, insofar as it relates contents of thought to themselves, while it positions model theory as straddling, as it were, intralinguistic proof-theoretic relations by accounting for the way cognition hooks up with the world in, paradigmatically, the perceptual and volitional moments of the reflex arc.

Proof-theoretic expressivist analyses of language remain a promising area of research, but some puzzles are raised by Incurvati and Schlöder's account of deontic modality. For the need to posit two distinct primitive attitudes (approval and disapproval) in order to account for the positive and negative strong modalities comes at the cost of a loss of the simplicity one expects in modal logic. And to date, the weak deontic modality of permission has not been examined in detail. In addition, as Incurvati and Schlöder point out, their analysis of deontic vocabulary is indirect (2019 p.747, and Forthcoming p.12): rather than expressing an action-guiding attitude directly, this analysis allows one to *infer* that such an attitude is expressed. While these issues do not scuttle the program, they do raise questions as to its sufficiency.

As a proof-theoretic logician, the bilateralist understands the speech act of rejection as one that relates a speaker to langauge; one rejects propositions, or claims, or interpreted sentences, on this account. Call this *linguistic rejection*. In other work (2021, Forthcoming), I develop a model-theoretic expressivist analysis of deontic modality. That analysis also employs a notion of rejection, but it is directed at choices rather than claims in a language. Call this *agentive rejection*. This model-theoretic account of deontic modality interprets representational or descriptive sentences, having a language-to-fit-world intentionality, in terms of possible worlds. Agentive rejection gives voice to action-guiding mental states having a practical or world-to-fit-language intentionality, and modelled by *deontic hyperplans* as maximally determinate plans of action. Deontic hyperplans specify what one would do as any person, at any place, at any time. In this regard, the moral frame of mind is understood as an exercise of one's practical rationality acrosss the membership of an in-principle limitless community (I shall discuss the artificiality of this notion of rejection, and of the possibility of identifying neurological analogues in processes of motor-representational neural mirroring and higher-order executive functioning).

Deontic hyperplans make use of two choice attitudes: single-mindedness and indifference. To choose to A single-mindedly is to *reject*, in the agentive sense, every choice incompatible with A. That is to say, the act of choosing single-mindedly is an act of self-government: one binds oneself to a course of action by refusing to allow oneself to do anything incompatible with it. The claim that one ought to A in C gives expression to the practical attitude of agentive rejection adopted toward every choice incompatible with doing A in C, which is modelled by a plan to single-mindedly choose to A in C. Just so, the claim that one ought not A in C expresses agentively rejecting choosing to A in C, modelled by the plan to single-mindedly choose not to A in C.

Unlike the speech act of rejection used in Incurvati and Schlöder's proof theory, this model-theoretic analysis of rejection not only provides a unified account of the positive and negative strong deontic modality, it also iterates in a way that naturally gives rise to the distinction between the strong and weak deontic modalities. To think that one is permitted to A in C is to *reject rejecting* doing A in C, modelled by the set of plans were one chooses *indifferently* whether or not to A in C (I am glossing over some of the formal details to be discussed in the talk). And indifference can in turn be defined in terms of single-mindedness: fix all of a hyperagent's single-minded choices across hyperspace, and she chooses indifferently to A at a point of choice just in case there is some action B that is incompatible with A which she could have undertaken without changing any of her single-minded choices. In this case, the choice between A and B is indifferent, and the agent rejects rejecting each of them. The result is a fully-compositional model-theoretic semantics for descriptive sentences, deontic sentences, and the Boolean operators (in Stovall 2021 I expand this semantics to model the individual and collective intentional modal operators *I shall* and *we shall*).

On the face of it, the proof-theoretic inferential expressivism of Incurvati and Schlöder is in competition with this model-theoretic hyperplan expressivism. For the latter allows for arbitrary embeddings, is recursively computable, and accomplishes this without multiplying primitive attitudes to account for different modalities (the epistemic modalities can be given a standard possible worlds analysis). Furthermore, the direct connection between moral claims and the attitudes expressed by them is naturally accounted for in terms of these practically intentional plans of action. For a plan is the very sort of thing that motivates one to act, and a deontic plan involves a kind of self-government characteristic of the moral law (and of rationality more generally).

In fact, I suspect these two points of view are compatible (this part of the research is ongoing). For if we make a distinction between the extension of a sentence, interpreted in terms of relations between *language* and *world* laid down by model-theoretic possible worlds and plans of action, and the comprehension or sense of a sentence interpreted in terms of *intralinguistic* relations laid down by proof-theoretic rules of inference, then it is open to interpret Incurvati and Schlöder's account as targeting the comprehension or sense of modal claims, while seeing my model-theoretic account as targeting their extensions. This has the appealing consequence of explaining that in virtue of which Incurvati and Schlöder's proof theory offers an indirect analysis of modal terminology, whereas mine is direct: for we can follow Frege and think of sense (in proof theory) as a mode of determination of reference (in model theory). Finally, the distinction between agentive and lingusitic rejection suggests the possibility of using the model-theoretical analysis of single-mindedness and *agentive* rejection as a basis for understanding the way the self-government characteristic of rational thought and agency is explicitly manifest in the exercise of *linguistic* rejection by speakers who are able to use the proof-theoretic methods of bilateral semantics.

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Combinatorial Proof Identity

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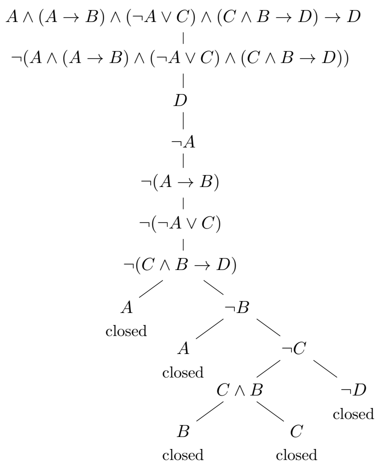
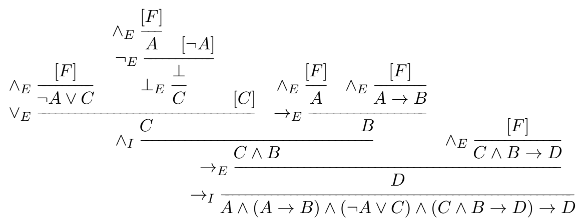
Proof theory is one of the pillars of mathematical logic. It is not only of interest for philosophers and theoretical computer scientists, it also has real life applications, as it provides the foundations for declarative programming languages and the formal verification of software. Yet, despite the crucial role played by formal proofs, we have no proper notion of proof identity telling us when two proofs are "the same". This is very different from other areas of mathematics, like group theory, where two groups are "the same" if they are isomorphic, or topology, where two spaces are "the same" if they are homeomorphic.

The problem is that proofs are usually presented by syntactic means, and depending on the chosen syntactic formalism, "the same" proof can look very different. Consider for example the following three "proofs" of the same formula $A \wedge (A \to B) \wedge (\neg A \vee C) \wedge (C \wedge B \to D) \to D$.





```
Goal A /\ (A -> B) /\ (~A \/ C) /\ (C /\ B -> D) -> D.
Proof.
intros h1. destruct h1 as [ha h2].
destruct h2 as [hab h3]. destruct h3 as [hac h4].
apply h4. split.
apply hab. exact ha.
destruct hac as [hna|hc]. elim hna. exact ha.
exact hc.
Qed.
```

On the left we have a tableaux derivation, on the right top a derivation in natural deduction, and on the right bottom a Coq script. Even though syntactically they all look very different from each other, we can argue that in principle they all do the same thing: they use A twice to justify B (via the subformula $A \to B$) and C (via the subformula $\neg A \vee C$), which in turn are then used to justify D (via the subformula $C \wedge B \to D$). This leads to the following natural question:

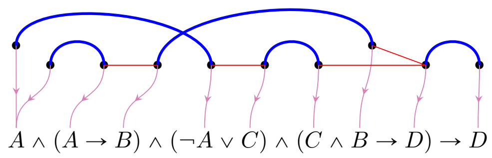
Is there notion of proof identity that puts this informal "sameness" on formal grounds?

The reason that sofar there is no satisfactory answer to this question is that, as observed in Straßburger (2019), at the current state of the art, *proof theory is not a theory of proofs but a theory of proof systems*. This means that the first step must be to find ways to describe proofs independently from the proof systems. In other words, we need a "syntax-free" presentation of proofs.

There are essentially two ways of giving syntax-independent presentations of formal proofs. First, we can define formal proofs via axioms that determine their properties. This is similar to what happens in algebra, where groups and rings are defined by simple axioms. In fact, this idea goes back to Lambek (1968) who defined proofs as morphisms in a category. Let us call this first approach the *axiomatic approach*. The second way is to define concrete mathematical objects that carry the meaning of proofs. This can, for example, be certain kinds of graphs, as they have been used in the form of proof nets by Girard (1987) or winning strategies in certain games as in Hyland and Ong (2000). Let us call this second approach the *combinatorial approach*. The ideal situation is of course when both approaches lead to equivalent notions of proof identity.

However, most category theoretical approaches to proof theory define the same notion of proof identity as the standard way of defining proof identity via proof normalization. For this reason we pursue here the second approach.

Combinatorial proofs, first introduced in Hughes (2006a), form a canonical proof presentation that (1) comes with a polynomial correctness criterion, (2) is independent of the syntax of proof formalisms (like sequent calculi, tableaux systems, resolution, Frege systems, or deep inference systems), and (3) can handle cut and substitution, and their elimination, as shown in Hughes (2006b) and Straßburger (2017). The example below shows the combinatorial proof corresponding to the three syntactic proofs shown above:



In a nutshell, a combinatorial proof consists of a cograph (shown in red/regular edges above), a purely *linear* proof (depicted above in blue/bold) and a part that corresponds to *contraction* and *weakening* (depicted above with purple/regular arrows). Combinatorial proofs can be composed horizontally and vertically, and can be substituted into each other.

The important observation to make here is that even though combinatorial proofs borrow some ideas from linear logic proof nets, they considerably depart from them, as they overcome the technical drawbacks that are usually associated to proof nets. In particular they can deal with logics that go beyond the linear realm, in particular, classical and intuitionistic logic.

In this presentation, I will give a gentle, easy accessible introduction to combinatorial proofs, and discuss whether they can help answering the question of the identity of proofs, as suggested in Hughes (2006b) and Straßburger (2019). I intend to make this presentation accessible to logicians with background from mathematics, philosophy, and computer science alike.

I will also briefly mention combinatorial proofs for other logics than classical propositional logics, in particular:

- relevance logics (as studied in Acclavio and Straßburger (2019a) and Ralph and Straßburger (2019)),
- intuitionistic logic (investigated in Heijltjes et al. (2019)),
- modal logics (as studied in Acclavio and Straßburger (2019b)), and
- first-order classical logic (studied in Hughes (2019) and Hughes et al. (2021)).

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Interpolation Without Equality

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Since the 1950s when Craig and Lyndon published the relevant papers, interpolation in classical predicate logic can be considered a traditional and well understood topic. Assuming that the set of logical connectives contains the symbols $\top$ and $\bot$ for truth and falsity, a succinct formulation of the interpolation theorem is as follows: *if φ and ψ are formulas such that $\varphi \to \psi$ is valid, then there exists a formula μ, called interpolant of φ and ψ, such that μ is built up from $\top$ and $\bot$ and from symbols that occur simultaneously in φ and ψ, and both $\varphi \to \mu$ and $\mu \to \varphi$ are valid*. In classical predicate logic, "valid" means *logically valid*, i.e. satisfied in all structures for the given language by all valuations of variables. However, the interpolation theorem makes sense (and may hold) also for other logics, both predicate and propositional. In those cases the meaning of "valid" is determined by the semantics of the given logic. In propositional logic(s), "symbols" (common to both φ and ψ) are just atoms, and the meaning of "valid" could be, for example, being satisfied at all nodes of all Kripke models.

In classical predicate logic and if a language L is fixed, a reasonable understanding of "symbols" is "free variables and the predicate and function symbols in L". For a set Γ of formulas, let $\mathrm{Rel}(\Gamma)$, $\mathrm{Fun}(\Gamma)$, and $\mathrm{FV}(\Gamma)$ denote the set of all predicate symbols that occur in Γ, the set of all function symbols that occur in Γ, and the set of all variables that occur free in Γ, respectively. We write $\mathrm{Rel}(\alpha)$, $\mathrm{Fun}(\alpha)$, and $\mathrm{FV}(\alpha)$ instead of $\mathrm{Rel}(\{\alpha\})$, $\mathrm{Fun}(\{\alpha\})$, and $\mathrm{FV}(\{\alpha\})$, and we also write e.g. $\mathrm{Rel}(\Gamma, \Delta)$ or $\mathrm{Rel}(\Gamma, \alpha)$ instead of $\mathrm{Rel}(\Gamma \cup \Delta)$ or $\mathrm{Rel}(\Gamma \cup \{\alpha\})$. With this notation, the requirements that an interpolant μ of φ and ψ is supposed to satisfy are (i) both $\varphi \to \mu$ and $\mu \to \varphi$ are logically valid, (ii) $\mathrm{Rel}(\mu) \subseteq \mathrm{Rel}(\varphi) \cap \mathrm{Rel}(\psi)$ and $\mathrm{FV}(\mu) \subseteq \mathrm{FV}(\varphi) \cap \mathrm{FV}(\psi)$, and (iii) $\mathrm{Fun}(\mu) \subseteq \mathrm{Fun}(\varphi) \cap \mathrm{Fun}(\psi)$.

Formulations of the interpolation theorem that appear in the literature often do not contain condition (iii) concerning function symbols. This is true also about (Lyndon, 1959a), as pointed out in (Motohashi, 1984). Motohashi in (1984) shows that (iii) can be achieved in classical predicate logic *with equality*.

A possible way of proving the interpolation theorem consists in finding a formulation for sequents and proving the claim by induction on the number of steps in a cut-free proof. The formulation for sequents is as follows. *Let $\Gamma, \Pi \Rightarrow \Delta, \Lambda$ be a logically valid sequent. Then there exists a formula μ such that (i) the two sequents $\Gamma \Rightarrow \Delta, \mu$ and $\Pi, \mu \Rightarrow \Lambda$ are logically valid, (ii) $\mathrm{Rel}(\mu) \subseteq \mathrm{Rel}(\Gamma, \Delta) \cap \mathrm{Rel}(\Pi, \Lambda)$ and $\mathrm{FV}(\mu) \subseteq \mathrm{FV}(\Gamma, \Delta) \cap \mathrm{FV}(\Pi, \Lambda)$, and (iii) $\mathrm{Fun}(\mu) \subseteq \mathrm{Fun}(\Gamma, \Delta) \cap \mathrm{Fun}(\Pi, \Lambda)$.* A proof of so formulated interpolation theorem is in Takeuti's book (1975) where, however, the condition (iii) is also missing. One might think that this is just a simplification to make the treatment more transparent. Indeed, the proof of the completeness theorem in (Takeuti, 1975) contains the sentence "in order to make the discussion simpler, we assume that there are no individual or function constants". However, while the completeness proof can be adjusted for language with function symbols, the problem with interpolation is deeper. For example, if the last step of a given cut-free proof of $\Gamma, \Pi \Rightarrow \Delta, \Lambda$ is $\forall$-left, the principal formula is $\forall z \alpha$ and it is in Π, then the last step is

$$\frac{\Gamma, \overbrace{\alpha_z(s), \Pi'} \Rightarrow \Delta, \Lambda}{\Gamma, \underbrace{\forall z \alpha, \Pi'} \Rightarrow \Delta, \Lambda}$$

where $\Pi = \Pi' \cup \forall z\alpha$ and the braces indicate that $\forall z\alpha$ belongs to Π and thus also $\alpha_z(s)$ is counted to the second set in the antecedent. In this situation, the "unsubstituted" term s and thus also an interpolant for the upper sequent may contain many function symbols that do not occur in the lower sequent, and it is not clear how to get rid of them when constructing an interpolant for the lower sequent.

We show that the full interpolation theorem (with condition (iii)) is true also for predicate logic *without equality*, and that its relatively easy consequence is the interpolation theorem for logic with equality (as stated in (Motohashi, 1984), but in order to make the discussion simpler, we do not distinguish positive and negative occurrences of symbols). We proceed along the same lines as in (Takeuti, 1975), but in the steps where function symbols cause problems we use the following claim, which can be called *enhanced generalization* lemma. *Assume that* $t_1, \ldots, t_n$ *are pairwise different terms such that the outermost symbol of each* t_i *has no occurrences, or has no free occurrences if it is a variable, in* $\Gamma \cup \Delta \cup \{\exists x_1 \ldots \exists x_n \varphi\}$. *Then, in predicate logic without equality, if* $\Gamma, \varphi_{x_1,\ldots,x_n}(t_1, \ldots, t_n) \Rightarrow \Delta$ *is a logically valid sequent, then also* $\Gamma, \exists x_1 \ldots \exists x_n \varphi \Rightarrow \Delta$ *is a logically valid sequent, and if* $\Gamma \Rightarrow \Delta, \varphi_{x_1,\ldots,x_n}(t_1, \ldots, t_n)$ *is a logically valid sequent, then also* $\Gamma \Rightarrow \Delta, \forall x_1 \ldots \forall x_n \varphi$ *is a logically valid sequent*. An interesting thing about this enhanced generalization is that while its validity for logic without equality in fact entails interpolation for both logics, it by itself *does not hold for logic with equality*.

The talk discusses work in progress: it seems plausible that the method of proving interpolation via enhanced generalization (or, *generalization over terms*) can be modified also for intuitionistic logic.

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The “Implications” of the Ultimate Subjectivism

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Implication is one of the most interesting logical constants with various incarnations, from classical and Heyting implications to many-valued, relevant and linear ones. The variety also includes some less well-known instances emerged to fulfil different philosophical (Okada, 1987), (Ruitenburg, 1991), algebraic (Celani-Jansana, 2001), (Celani-Jansana, 2005), relational (Litak-Visser, 2018), and proof-theoretical (Visser, 1981), (Visser, 2002), (Iemhoff, 2003) motives. In this talk we will unify this variety by introducing the most general formalization of the notion of implication by certain algebraic apparatus. Then, we move to the philosophical side to present a generalized and dynamic reading of intuitionism as the philosophy of the *ultimate subjectivism* in which truth heavily depends not only on the creative subject's mental world but also on the *process* of her introspections and the *temporal* nature of reasoning. We will formalize this type of intuitionism by temporal quantales that we call *spacetimes*. We will show that spacetimes are powerful enough to represent any abstract implication. More provocatively, this implies that when it comes to implications, any world can be seen as a radically subjective world. We will then introduce the logic of spacetimes and present the corresponding soundness-completeness theorems to unify the realm of sub-structural and sub-intuitionistic logics.

In the rest of this abstract, let us expand more on the previous line of ideas. First, to specify what we mean by generalized intutionism and its spatio-temporal nature, we have to start with its core idea that a proposition is not just a truth assignment to the mental states but a process to check the truth value that may change the very state it is supposed to observe. Here, conjunction is not just the meet of the truth values, but the composition of the corresponding processes and true is not just the top element (if there is any) but the trivial process that outputs true for all the states and leaves the states intact. Therefore, we formalize the world of propositions by a monoidal poset $(A, \leq, \otimes, e)$ in which A is the set of all propositions, $\leq$ is its provability order, $\otimes$ is the conjunction and e as the value *true*. In some cases, we can even go further to use topology and its linear version, i.e., the quantales to demand that propositions must be also *finitely verifiable* meaning that their processes use only finite amount of information of the mental states. Hence, we can read quantales as the world of intuitionistic propositions that even introspection (observation of the truth in a mental state) changes the state we are currently in, see (Abramsky-Vickers, 1993). Finally, to complete the pure spatial picture with some temporal spice, we use the temporal modality “*happened at some point in the past*” and formalize it by a join-preserving operator $\nabla : \mathcal{X} \to \mathcal{X}$. It must be join-preserving as it is existential in nature and hence commutes with all the possible disjunctions. We call the pair $(\mathcal{X}, \nabla)$ a spacetime and we will use its temporal modality to capture the temporal nature of reasoning, as we will see in a moment.

Coming back to implications, we know that in any logical discourse, implications are the internalizers of the provability structure of that discourse. There are many different structures that we can expect an implication to internalize. The minimum property of the provability order is its reflexivity, i.e., $\phi \vdash \phi$ and its transitivity, i.e., “$\phi \vdash \psi$ *and* $\psi \vdash \theta$ *implies* $\phi \vdash \theta$”. Therefore, reading $\mathcal{A} = (A, \leq, \otimes, e)$ as the formalization of the world of propositions, by an implication on $\mathcal{A}$, denoted by the symbol $\to$, we mean a function from $(A, \leq)^{op} \times (A, \leq)$ to $(A, \leq)$ such that it is order-preserving in its both arguments and:

- (internalized reflexivity) $e \leq a \to a$,
- (internalized transitivity) $(a \to b) \otimes (b \to c) \leq (a \to c)$.

As it is well-known any quantale is powerful enough to internalize its own structure by its substructural implication $\Rightarrow$ defined by $a \Rightarrow b = \bigvee\{c \in \mathcal{X} \mid a \otimes c \leq b\}$. In a similar way, any spacetime $\mathcal{S} = (\mathcal{X}, \nabla)$ has its own implication defined by $a \to_{\mathcal{S}} b = \bigvee\{c \in \mathcal{X} \mid a \otimes \nabla c \leq b\}$. This implies that $c \leq a \to_{\mathcal{S}} b$ iff $a \otimes \nabla c \leq b$, formalizing the temporal situation that $a \to_{\mathcal{S}} b$ is provable from the assumption c iff b is provable by a assuming that we have already proved c. One of the main aims of this talk is to show that any abstract implication is representable as an implication of a suitable spacetime (up to a correction term):

Theorem 1. *(Representation Theorem I) Let $\mathcal{A} = (A, \leq, \otimes, e)$ be a monoidal poset and $\to$ be an implication over $\mathcal{A}$. Then, there exists a spacetime $\mathcal{S} = (\mathcal{X}, \nabla)$, a monotone map $F : \mathcal{X} \to \mathcal{X}$ (correction term) and a monoidal embedding $i : \mathcal{A} \to \mathcal{X}$ such that $i(a \to_{\mathcal{A}} b) = F(i(a)) \to_{\mathcal{S}} F(i(b))$.*

One may aim to eliminate the correction term F. In the general case, this is impossible. However, for some stronger implications the term becomes redundant. More precisely, if $\mathcal{A} = (A, \leq, \otimes, e)$ has the left residuation $\Rightarrow$, i.e., the binary operator for which we have $a \otimes b \leq c$ iff $b \leq a \Rightarrow c$, for any $a, b, c \in A$, then we say an implication on $\mathcal{A}$ internalizes the closed monoidal structure of $\mathcal{A}$, if for all $a, b, c \in A$:

$$a \to b \leq c \otimes a \to c \otimes b$$

and

$$a \otimes b \to c \leq b \to (a \Rightarrow c)$$

Theorem 2. *(Representation Theorem II) Let $\mathcal{A} = (A, \leq, \otimes, e)$ be a monoidal poset and $\to$ be an implication over $\mathcal{A}$ that internalizes its closed monoidal structure. Then, there exists a spacetime $\mathcal{S} = (\mathcal{X}, \nabla)$ and a monoidal embedding $i : \mathcal{A} \to \mathcal{X}$ mapping $\to$ to $\to_{\mathcal{S}}$.*

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Affine Spatial Logics: Three Dimensions and Beyond

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Spatial logic can be viewed as a formal language with geometrical interpretation, where variables range over geometrical entities and relation and function symbols are interpreted as geometrical relations and functions. In recent years, there has been a considerable development within the topological spatial logics based on regions, rather than points (see Aielo et al. (2007)). The motivation came from the research on qualitative spatial reasoning, a more practically oriented subfield of AI/KR&R. The field saw a rapid development of results concerning topological logics. We focus on another branch of region-based spatial logics dealing with affine geometry. Recall that an affine transformation maps straight lines to straight lines, preserves parallelism and ratios of lengths along parallel straight lines. This geometry is often — somehow misleadingly — described as Euclidean geometry without distance and thus is an important area of both practical and theoretical investigations.The research on formalising parts of this geometry in logic is, however, scarce: only a handful of papers investigate such systems, mostly in the case of the real plane. In particular, an effort has been made to axiomatise one such logic and explore its expressive power. It turned out that when limiting what counts as regions in such logic to rational open polygons, one can harness the relative high expressiveness of this logic and construct formulas describing lines and various relations among them, allowing an introduction of coordinate frames and, by means of these, fixing of the all the regions in reference to a given frame (see Davis et al. (1999) and Pratt (1999)). The resulting axiomatisation relies heavily on this result (see Trybus (2016)). In this paper we describe work intended to mimic these results in the case of the three-dimensions. We work with the following class of structures. Let $\mathfrak{M}^n = \langle ROQ(\mathbb{R}^n), \mathfrak{conv}^{\mathfrak{M}}, \leq^{\mathfrak{M}} \rangle$, where:

$$\leq^{\mathfrak{M}} = \{\langle a, b\rangle \in ROQ(\mathbb{R}^n) \times ROQ(\mathbb{R}^n) \mid a \subseteq b\}$$

and

$$\mathfrak{conv}^{\mathfrak{M}} = \{a \in ROQ(\mathbb{R}^n) \mid a \text{ is convex.}\}$$

In the above, ROQ stands for 'regular open rational polygons' and is a way of choosing appropriate region candidates (but see Lando and Scott (2019)). We focus on $n = 3$ but we start by showing that logics of different dimensionalities must have different theories, thus justifying our work on higher dimensions. The proof consists in defining a formula representing a particular case described by means of Helly's theorem, a standard result in convex geometry. We then move on the three-dimensional case exploring the expressiveness of this logic and consequently making the first step of showing that talking about coordinate frames is indeed possible there.

We show that the notion of a half-space is expressible and thus, indirectly we are able to talk about planes (and their relations, such as parallelism) as well. We then consider various arrangements of planes in the three-dimensional space:

(i) a sheaf: where all the planes meet in a single line;

(ii) a prism: where two of the planes meet in a line not on the third plane and meet the third plane in two separate, parallel lines;

(iii) a corner: where two of the planes meet the thid plane in two separate, non-parallel lines and meet each other in a line that passes through the third plane.

Since in all the above cases, the number of domains into which the entire space is being partitioned changes (6 domains for a sheaf, 7 for a prism and 8 for a corner) and it can be expressed in terms of products of respective half-spaces or their complements, one can build formulas describing all three cases in $\mathfrak{M}^3$. Note that (iii) can be used as a basis for a coordinate frame. We show how to express various required notions, including the units of measurement and fundamental operations of addition and multiplication (see Bennett (1995)) in all the planes forming the coordinate frame (we emulate a two-dimensional result in three dimensions). Thus, the stage is set for all the expressivity results regarding coordinate frames to be carried over from Trybus (2016) and Pratt (1999), including the construction of two-dimensional fixing formulas. However, it is not obvious how to proceed with constructing three-dimensional fixing formulas required in $\mathfrak{M}^3$ and even once this is accomplished, building an axiom system is not just a simple case of extending the two-dimensional results. We discuss potential axiom candidates and the challengeas ahead. We also talk about the possibility of extending the work to dimensions greater than three.

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Admissible Rules in Intuitionistic Modal Logic

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Proof-theoretic research of logical systems is usually concerned with axiomatisation and derivation. Axioms and derivation rules determine the theorems of the logic. The aim is to find a minimal set of axioms and rules that define the logic. An interesting question is: what is the maximal set of inference rules? Or in other words, given a logic, which rules can be added without changing the set of theorems of the logic? These rules are called the admissible rules of the logic. Admissible rules are interesting to study because they form an invariant for the logic independently from the chosen axiomatisation. Thereby they give insight in the structure of all possible inferences in a logic. We will see that the field of admissible rules is an interesting and challenging area in proof theory.

The history of admissible rules started in the second half of the last century and has a large Czech component. The research got a boost in 1975 with one of Friedman's problems (Friedman, 1975): Is admissibility in IPC decidable? The question was positively answered by Rybakov, who published a series of papers showing that admissibility in IPC, many intermediate logics, and many modal logics above K4 is decidable, see (Rybakov, 1997). Later, full descriptions of the admissible rules are established in terms of a basis for many of these logics. A basis is a set of admissible rules that derive all other admissible rules in the logic. The so-called Visser rules form a basis for the admissible rules for IPC, independently shown by Rozière (1992) and Iemhoff (2001). From here, the Czech story begins with a series of papers by Jeřábek. In 2005, he constructed modal Visser rules forming bases for the admissible rules of classical modal logics (Jeřábek, 2005). After that multiple papers appeared about descriptions, bases, and complexity of the admissible rules for modal logic and Łukasiewics logic.

Given the work in IPC and classical modal logic, we are interested in another broad range of logic: intuitionistic modal logic. This is a big project, because intuitionistic modal logics can be defined in different ways. We focus on intuitionistic modal logics only containing the $\Box$, and without a $\Diamond$. We ask ourselves the following question. Can the methods and results for IPC and classical modal logic be combined to obtain admissibility results for these intuitionistic modal logics?

The answer is yes! We found that a natural combination of the admissible rules for IPC and classical modal logics form the admissible rules for six interesting intuitionistic modal logics. These six logics are: iCK4, iCS4 $\equiv$ IPC, strong Löb logic iSL, modalized Heyting calculus mHC, Kuznetsov-Muravitsky logic KM and propositional lax logic PLL. The crucial axiom of these logics is the completeness axiom, $A \to \Box A$. In Kripke semantics, this corresponds to frames equipped with a partial order and a modal relation with the strong condition that the modal relation is contained in the partial order. Although they form a small set of logics, almost all have interesting interpretations. Logics iSL, mHC, and KM have close connections to provability logic, see respectively the work of Ardeshir and Mojtahedi (2018), Esakia (2006), and Muravitsky (2014). Logic PLL is a little bit different from the other five logics, containing a modality with flavors from both $\Box$ and $\Diamond$. It has many interesting application in algebra, topos theory and hardware verification (Fairtlough and Mendler, 1997).

We establish a full description of the admissible rules of these logics. We describe the admissible rules in terms of a proof system using the same strategy from Iemhoff and Metcalfe

(2009). They provide Gentzen-style proof systems for admissibility for IPC and several modal logics above K4. We combine these systems into a system for admissibility of the intuitionistic modal logics that we study. In contrast to well-known proof systems for logics that reason about formulas or sequents, these admissibility proof systems reason about rules. In other words, they contain rules about rules.

We obtain a basis for the admissible rules for each of the considered logics. The bases contain Visser-like rules that form a natural fusion of the Visser rules for IPC and the modal Visser rules. We extract these bases from our constructed admissibility proof systems. A way to think about the Visser-like rules is that they reflect the structure of extensions of the corresponding Kripke models. The strong condition on the models forced by the completeness axiom plays a crucial role here. In this way, each considered logic has different, but related, Visser rules.

Finally, a big advantage of the admissibility proof systems is that decidability of admissibility immediately follows from the decidability of the logic. Thereby we positively answer Friedman's question, but now for the considered intuitionistic modal logics.

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Choice-Free Duality for Ortholattices by Means of Spectral Spaces

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Introduction The existing topological representation depends upon Alexander's Subbase Theorem, which is equivalent to the Boolean Prime Ideal Theorem. In the present work, we give a choice-free topological representation of ortholattices by spectral spaces. We then introduce a new subclass of spectral spaces which we call *upper Vietoris orthospaces* to characterize the duals of ortholattices under our representation theorem. It is then shown how our constructions give rise to a choice-free dual equivalence of categories between the category of ortholattices and the category of upper Vietoris orthospaces. Our duality combines Bezhanishvili and Holliday's (2020) choice-free spectral space representation for Boolean algebras with Goldblatt's (1975) and Bimb's (2007) choice-dependent Stone space representation for ortholattices.

Choice-free duals of ortholattices Consider an arbitrary ortholattice L. Let X_L^+ be the set of proper filters of L. We equip X_L^+ with the topology generated by the sets of the form $\widehat{a}$, where $\widehat{a} := \{u \in X_L^+ \mid a \in u\}$ for $a \in L$. We may define a binary relation $\perp$ on X_L^+ by requiring that $u \perp v$ if and only if there exists $a \in u$ with $a^\perp \in v$.

Definition. The *choice-free dual* of an ortholattice L, also denoted X_L^+, is the expansion of the topological space X_L^+ by the binary relation $\perp$ defined above.

Definition. A subset S of a relational structure $(X, \perp)$ is $\perp$-*regular* if $S^{\perp\perp} = S$, where $S^\perp := \{x \in X \mid \forall y \in X\, x \perp y\}$.

Theorem. The ortholattice $\mathscr{COR}(X_L^+)$ of compact open $\perp$-regular subsets of X_L^+ is L.

Characterization of UVO-spaces The duals of distributive lattices under Priestley duality is characterized by compactness and the Priestley separation axiom. It is arguably useful to have a similar characterization for the choice-free duals of ortholattices as well.

Definition. An expansion $(X, \perp)$ of a topological space X by an irreflexive symmetric binary relation $\perp$ on X is an *upper Vietoris orthospace*, or a *UVO-space*, if and only if:

1. X is T_0.
2. $\mathscr{COR}(X)$ is closed under $\cap$ and ${}^\perp$.
3. $\mathscr{COR}(X)$ is a basis of X.
4. Every proper filter of $\mathscr{COR}(X)$ is of the form $\mathscr{COR}^X(x)$ for some $x \in X$, where $\mathscr{COR}^X(x) := \{U \in \mathscr{COR}(X) \mid x \in U\}$.
5. If $x \perp y$, then there is $U \in \mathscr{COR}(X)$ such that $x \in U$ and $y \in U^\perp$.

Theorem. The choice-free dual of L is a UVO-space; conversely, for every UVO-space $(X, \perp)$, there is an ortholattice L whose choice-free dual is topologically isomorphic to $(X, \perp)$. *A fortiori*, for an ortholattice L, the choice-free dual of L is spectral.

Category of UVO-spaces The aforementioned one-to-one correspondence between ortholattices and UVO-spaces can be made into a dual categorical equivalence.

Definition. A function $f : (X, \perp) \to (X', \perp')$ between UVO-spaces is a *UVO-map* if it is spectral and p-morphic with respect to $\not\perp$ and $\not\perp'$.

Theorem. The category **UVO** of UVO-spaces and UVO-maps is dually equivalent to the category **OrthLatt** of ortholattices.

Duality dictionary One can now translate lattice theoretic notions into topological language. This is summarized in Table 1, subject to the Definition below.

Definition.
1. A UVO-space is *complete* if $U^{\perp\circ\perp} \in \mathscr{COR}(X)$ for every open set $U \subseteq X$.
2. X_{iso} is the set of isolated points of X.
3. A *UVO-embedding* is an injective UVO-map $f : X \to Y$ such that for every $U \in \mathscr{COR}(X)$, there exists some $V \in \mathscr{COR}(Y)$ such that $f[U] = f[X] \cap V$.
4. If X and Y are UVO-spaces, then their *UVO-sum* $X+Y$ is the space whose underlying carrier set is $X+Y := X \cup Y \cup (X \times Y)$ and whose topology is generated by sets of the form $U \cup V \cup (U \times V)$ for $U \in \mathscr{COR}(X)$ and $V \in \mathscr{COR}(Y)$, together with the binary relation $\perp_{X+Y}$, which is defined as the symmetric closure of:
$$\perp_X \cup \perp_Y \cup (X \times Y) \cup \{\langle\langle x,y\rangle, x'\rangle \mid x \perp_X x'\} \cup \{\langle\langle x,y\rangle, y'\rangle \mid y \perp_Y y'\} \\ \cup \{\langle x,y\rangle, \langle x',y'\rangle \mid x \perp_X x', y \perp_Y y'\}.$$
5. A point $u \in X$ is *principal* if there exists an open neighborhood U of u such that $v \notin U$ for all v strictly below u in the specialization preorder. $\mathfrak{P}(X)$ is the set of such points.
6. $\mathscr{R}(X, \perp)$ is the set of $\perp$-regular subsets of X.

OrthLatt	**UVO**
complete lattice	complete UVO-space
atom	isolated point
atomic lattice	$\mathrm{Cl}(X_{\text{iso}}) = X$
injective homomorphism	surjective UVO-map
surjective homomorphism	UVO-embedding
subalgebra	image under UVO-map
direct product	UVO-sum
MacNeille completion	$\mathscr{R}(\mathfrak{P}(X))$
canonical extension	$\mathscr{R}(X)$

Table 1: Duality dictionary for **Orthlatt** and **UVO**

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